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SOME THEOREMS ON GENERALISED MEIJER TRANSFORM

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Abstract.

In this paper two theorems on generalised Meijer transform due to Banerjee [1, p.433] defined by

$$\varphi_1(p)=\int_0^\infty (2px)^{m-\frac{1}{2}} e^{-\frac{1}{2}pxn} \varphi(a,c;\ 2px) f(x)dx,$$

have been proved. Some interesting integrals involving hyper-geometric functions are evaluated by the application of theorems.

1. Introduction.

The Laplace transform of a functin $f\in L(0,\infty)$ is defined by

$$\phi(p)=\int_0^\infty e^{-px} f(x)dx. \tag{1.1}$$

Meijer [4, p.727] generalised it in the form:

$$\varphi(p)=\int_0^\infty (px)^{-k-\frac{1}{2}} e^{-\frac{1}{2}px} W_{k+\frac{1}{2},m}(px) f(x)dx \tag{1.2}$$

When $k=m$, (1.2) reduces to (1.1) by virtue of the well known identity

$$W_{m+\frac{1}{2},m}(x)=e^{-\frac{1}{2}x}\, x^{m+\frac{1}{2}}$$

Banerjee [1, p.433] has introduced two generalisations which he calls as generalised Meijer transform of first and second kind. The first generalisation is given by

$$\varphi_1(p)=\int_0^\infty (2px)^{m-\frac{1}{2}} e^{-\frac{1}{2}pxn} \varphi(a,c;2px) f(x)dx \tag{1.3}$$

where $\qquad \varphi(a,c;x)={}_1F_1(a;c;x)$

The second generalisation has been defined similarly with ϕ in place of φ, where

$$\psi(a,c;x)=\frac{\Gamma(1-c)}{\Gamma(a-c+1)}\varphi(a,c;x)+\frac{\Gamma(c-1)}{\Gamma(a)}x^{1-c}\varphi(a-c+1,2-c;x) \tag{1.4}$$

provided that $\mathrm{Re}(a)>0$, $\mathrm{Re}(c)>0$, $\mathrm{Re}(n)>0$, and $f\in L(0,\infty)$.

The object of this paper is to establish two theorems on generalised Meijer transform of the first kind defined by (1.3). The theorems have been illustrated by means of some suitable examples, so as to give certain infinite integrals involving hypergeometric functions.

Throughout this note, we shall represent(1.1), (1.2) and (1.3) symbolically as

$$\psi(p)\doteqdot f(x)\ ;\ \ \varphi(p)\ \frac{k+\frac{1}{2}}{m}\ f(x)\ \text{ and }\ \varphi_1(p)\frac{n,m}{a,c}\ f(x)\ \text{ respectively.}$$

2. THEOREM I.

If $f(p)\doteqdot\psi(x)$ *and* $\varphi_1(p)\ \frac{n,m}{a,c}\ f(x)$

then

$$\varphi_1(p)=(2p)^{m-\frac{1}{2}}\Gamma\left(m+\frac{1}{2}\right)\int_0^\infty\left(\frac{pn}{2}+x\right)^{-m-\frac{1}{2}}{}_2F_1\left\{a,m+\frac{1}{2};c;\frac{4p}{np+2x}\right\}\psi(x)dx, \tag{2.1}$$

provided that f *and* ψ *both* $\in L(0,\infty)$, $\mathrm{Re}\ (a)>0$, $\mathrm{Re}(c)>0$, $\mathrm{Re}(n)>0$, $\mathrm{Re}\left(m+\frac{1}{2}\right)>0$, $\mathrm{Re}(p)>0$.

PROOF. We have

$$\varphi_1(p)=\int_0^\infty(2px)^{m-\frac{1}{2}}e^{-\frac{1}{2}pxn}{}_1F_1\{a;c;2px\}f(x)dx$$

$$=\int_0^\infty(2px)^{m-\frac{1}{2}}e^{-\frac{1}{2}pxn}{}_1F_1\{a;c;2px\}\left[\int_0^\infty e^{-ux}\psi(u)du\right]dx.$$

$$=\int_0^\infty(2p)^{m-\frac{1}{2}}\psi(u)\left[\int_0^\infty x^{m-\frac{1}{2}}e^{-x\left(\frac{np}{2}+u\right)}{}_1F_1\{a;c;2px\}\,dx\right]du$$

$$=(2p)^{m-\frac{1}{2}}\Gamma\left(m+\frac{1}{2}\right)\int_0^\infty\left(\frac{np}{2}+u\right)^{-m-\frac{1}{2}}{}_2F_1\left\{a,m+\frac{1}{2};c;\frac{4p}{np+2u}\right\}\psi(u)du$$

On changing the order of integration which is justified under the conditions stated with the theorem by virtue of the Fubini's theorem and evaluating the inner integral by formula [3, p.220].

$$\int_0^\infty e^{-pt} t^{\sigma-1} {}_1F_1(a;c;\lambda t)dt = p^{-\sigma}\Gamma(\sigma)\,{}_2F_1\left(a,\sigma;c;\frac{\lambda}{p}\right) \tag{2.2}$$

where $\mathrm{Re}(\sigma)>0$.

Applications.

EXAMPLE (i) If we take the operational pair [3, p. 238]

$$\phi(t)=\frac{t^{\rho-1}}{\Gamma(\rho)}$$
$$=p^{-\rho}=f(p) \tag{2.3}$$

then

$$\varphi_1(p)=2^{2m-\rho}\, p^{\rho-1}\, n^{\rho-m-\frac{1}{2}}\,\Gamma\left(m-\rho+\frac{1}{2}\right){}_2F_1\left\{a,m-\rho+\frac{1}{2},c;\frac{4}{n}\right\} \tag{2.4}$$

Putting these values of $\phi(t)$ and $\varphi_1(p)$ in the result (2.1), we obtain

$$\int_0^\infty (np+2x)^{-m-\frac{1}{2}} x^{\rho-1}\,{}_2F_1\left\{a,m-\rho+\frac{1}{2};c;\frac{4p}{np+2x}\right\}dx$$

$$=\frac{2^{-\rho}(np)^{\rho-m-\frac{1}{2}}\Gamma(\rho)\Gamma\left(m-\rho+\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)}\,{}_2F_1\left\{a,m-\rho+\frac{1}{2};c;\frac{4}{n}\right\} \tag{2.5}$$

provided that $\mathrm{Re}(\rho)>0$ and $\mathrm{Re}\left(m-\rho+\frac{1}{2}\right)>0$.

EXAMPLE (ii)

Take [3, p. 295]

$$\phi(x)=\alpha^{\mu}\beta^{\mu}x^{-k-\lambda}(\alpha+x)^{k-\mu}(\beta+x)^{\lambda-\mu}\,{}_2F_1\left\{\mu-k,\mu-\lambda;1-k-\lambda;\frac{x(\alpha+\beta+x)}{(\alpha+x)(\beta+x)}\right\}$$
$$\doteqdot\Gamma(1-k-\lambda)p^{-1}e^{\frac{1}{2}(\alpha+\beta)p}\,W_{k,\mu-\frac{1}{2}}(\alpha p)W_{\lambda,\mu-\frac{1}{2}}(\beta p)=f(p) \tag{2.6}$$

then [3, p. 216]

$$\varphi_1(p)=\Gamma(1-k-\lambda)(2p)^{m-\frac{1}{2}}$$
$$\times\sum_{\mu_1,-\mu_1}\sum_{\upsilon_1,-\upsilon_1}\frac{\Gamma(-2\mu_1)\Gamma(-2\upsilon_1)\Gamma\left(m+\mu_1+\upsilon_1+\frac{1}{2}\right)\alpha^{\mu_1+\frac{1}{2}}\,\beta^{\upsilon_1+\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-\mu_1-k\right)\Gamma\left(\frac{1}{2}-\upsilon_1-\lambda\right)\left(\frac{1}{2}np\right)^{m+\mu_1+\upsilon_1+\frac{1}{2}}}$$
$$\times F_A\left\{m+\mu_1+\upsilon_1+\frac{1}{2}:\mu_1-k+\frac{1}{2},\ \upsilon_1-\lambda+\frac{1}{2},\ a;2\mu_1+1,\ 2\upsilon_1+1,\ c;\right.$$
$$\left.\frac{2\alpha}{np},\ \frac{2\beta}{np},\ \frac{4}{n}\right\} \tag{2.7}$$

by virtue of the relation

$$ {}_1F_1\left(\frac{1}{2}+\mu-k;2\mu+1;z\right)=z^{-\mu-\frac{1}{2}}e^{\frac{1}{2}z}\ M_{k,\mu}(z) \tag{2.8}$$

Using the above values of $\phi(t)$ and $\varphi_1(p)$ in the result (2.1) we obtain after a little simplification.

$$\int_0^\infty x^{-k-\lambda}(\alpha+x)^{k-\mu}(\beta+x)^{\lambda-\mu}(np+2x)^{-m-\frac{1}{2}}\,{}_2F_1\left\{a, m+\frac{1}{2} : c; \frac{4p}{np+2x}\right\}$$

$$\times {}_2F_1\left\{\mu-k, \mu-\lambda; 1-k-\lambda; \frac{x(\alpha+\beta+x)}{(\alpha+x)(\beta+x)}\right\}dx$$

$$=\frac{\Gamma(1-k-\lambda)}{\Gamma\left(m+\frac{1}{2}\right)}\sum_{\mu_1,-\mu_1}\sum_{\upsilon_1,-\upsilon_1}\frac{\Gamma(-2\mu_1)\Gamma(-2\upsilon_1)\Gamma\left(\mu_1+\upsilon_1+m+\frac{1}{2}\right)\alpha^{\mu_1-\mu+\frac{1}{2}}\beta^{\upsilon_1-\mu+\frac{1}{2}}}{\Gamma(1-\mu_1-k)\Gamma(1-\upsilon_1-\lambda)\left(\frac{1}{2}np\right)^{\mu_1+\upsilon_1+m+\frac{1}{2}}}$$

$$\times F_A\left\{\mu_1+\upsilon_1+m+\frac{1}{2} : \mu_1-k+\frac{1}{2}, \upsilon_1-\lambda+\frac{1}{2}, a; 2\mu_1+1,\right.$$

$$\left. 2\upsilon_1+1, c; \frac{2\alpha}{np}, \frac{2\beta}{np}, \frac{4}{n}\right\} \tag{2.9}$$

provided that $\mathrm{Re}(1-k-\lambda)>0$, $\mathrm{Re}\left(m\pm 2\mu-\frac{1}{2}\right)>0$, $\mathrm{Re}(\alpha+\beta)>0$,

$\mathrm{Re}(4p+\alpha+\beta)>0$, $|\arg\alpha|<\pi$, $|\arg\beta|<\pi$,

where $\sum_{\mu_1,-\mu_1}$ or $\sum_{\upsilon_1,-\upsilon_1}$ indicates $\left(\mu_1=\mu-\frac{1}{2}=\upsilon_1\right)$ that to the expression following it a similar expression is to be added after writing $-\mu_1$ for μ_1 or $-\upsilon_1$ for υ_1.

3. THEOREM II.

If $$\varphi_1(p)\ \underset{a_1, c_1}{\overset{n_1, m_1}{=\!=}}\ f(x)$$

and $$f(p)\ \underset{a_2, c_2}{\overset{n_2, m_2}{=\!=}}\ \phi(x)$$

then

$$\varphi_1(p)=2^{2m_1+2m_2}p^{m_1-\frac{1}{2}}\Gamma(m_1+m_2)$$

$$\times\int_0^\infty x^{m_2-\frac{1}{2}}(n_1p+n_2x)^{-m_1-m_2}F_2\left\{m_1+m_2; a_1, a_2; c_1, c_2; \frac{4p}{n_1p+n_2x}, \frac{4x}{n_1p+n_2x}\right\}\phi(x)dx \tag{3.1}$$

provided that both f *and* $\phi\in L(0,\infty)$, $\mathrm{Re}(m_1+m_2)>0$.

PROOF. We have

$$\varphi_1(p)=\int_0^\infty (2px)^{m_1-\frac{1}{2}} e^{-\frac{1}{2}npx} {}_1F_1(a_1;c_1;2px)\Big\{\int_0^\infty (2ux)^{m_2-\frac{1}{2}} e^{-\frac{1}{2}n_2ux}$$

$$\times {}_1F_1(a_2;c_2;2ux)\phi(u)du\Big\}dx$$

$$=2^{m_1+m_2-1} p^{m_1-\frac{1}{2}} \int_0^\infty u^{m_2-\frac{1}{2}} \phi(u)\Big\{\int_0^\infty x^{m_1+m_2-1} e^{-\frac{1}{2}x(n_1p+n_2u)}$$

$$\times {}_1F_1(a_1;c_1;2px) {}_1F_1(a_2;c_2;2ux)dx\Big\}du.$$

The inner integral when evaluated by [3, p. 216] gives the required result.

EXAMPLE.

Let $\phi(x)=x^{-\rho}$

$$\frac{n_2, m_2}{a_2, c_2} \; 2^{2m_2-\rho} p^{\rho-1} n_2^{\rho-m_2-\frac{1}{2}} \Gamma\Big(m_2-\rho+\frac{1}{2}\Big) {}_2F_1\Big\{a_2, m_2-\rho+\frac{1}{2} : c_2; \frac{4}{n_2}\Big\}$$

$$=f(p) \tag{3.2}$$

then

$$\varphi_1(p)=2^{2\left(m_1+m_2-\frac{1}{2}\right)} p^{-1} n_1^{-m_1-\rho+\frac{1}{2}} n_2^{-m_2+\rho-\frac{1}{2}} \Gamma\Big(m_1+\rho+\frac{1}{2}\Big)\Gamma\Big(m_2-\rho+\frac{1}{2}\Big)$$

$$\times {}_2F_1\Big(a_1, m_1+\rho-\frac{1}{2} : c_1 : \frac{4}{n_1}\Big) {}_2F_1\Big(a_2, m_2-\rho+\frac{1}{2} : c_2; \frac{4}{n_2}\Big) \tag{3.3}$$

Putting these values of $\phi(x)$ and $\varphi_1(p)$ in [3.1] and simplifying we get

$$\int_0^\infty x^{m_2-\rho}(n_1p+n_2x)^{-m_1-m_2} F_2\Big\{m_1+m_2; a_1, a_2; c_1, c_2; \frac{4p}{n_1p+n_2x}, \; \frac{4x}{n_1p+n_2x}\Big\}dx$$

$$=\frac{\Gamma\Big(m_1+\rho-\frac{1}{2}\Big)\Gamma\Big(m_2-\rho+\frac{1}{2}\Big)}{\Gamma(m_1+m_2)} p^{m_2-\frac{1}{2}} n_1^{-m_1-\rho+\frac{1}{2}} n_2^{-m_2+\rho-\frac{1}{2}}$$

$$\times {}_2F_1\Big(a_1, m_1+\rho-\frac{1}{2} : c_1; \frac{4}{n_1}\Big) \; {}_2F_1\Big(a_2, m_2-\rho+\frac{1}{2} : c_2; \frac{4}{n_2}\Big)$$

$$\mathrm{Re}\Big(m_1+\rho-\frac{1}{2}\Big)>0, \quad \mathrm{Re}\Big(m_2-\rho+\frac{1}{2}\Big)>0.$$

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