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ON RIEMANNIAN SPACES WITH PARALLEL WEYL'S PROJECTIVE CURVATURE TENSOR

By Hidehiro Takeda and Yoshiyuki Watanabe

§1. Introduction.

It is well known that a Riemannian space with vanishing Weyl's projective curvature tensor $W_{kji}{}^h$ is a space of constant curvature and is necessarily locally symmetric.

In §2, we shall prove that a Riemannian space with parallel Weyl's projective curvature tensor is locally symmetric.

On the other hand, it is well known that the holomorphically projective curvature tensor $P_{kji}{}^h$ of a Kählerian space ([2]) which is invariant under any holomorphically projective transformation correspondents to the Weyl's projective curvature tensor of a Riemannian space and will be called H-projective curvature tensor. It is well known that a Kählerian space with vanishing H-projective curvature tensor is a space of constant holomorphic curvature ([2]) and is necessarily locally symmetric.

In §3, we shall prove that a Kählerian space with parallel H-projective curvature tensor is locally symmetric.

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§2. Riemannian spaces with parallel Weyl's projective curvature tensor.

Let M be a n-dimensional $(n>2)$ Riemannian space and $\{x^i\}$ a locally coordinate system. As usual, g_{ji}, $R_{kji}{}^h$ and $R_{ji}=R_{kji}{}^k$ denote the Riemannian metric tensor, the curvature tensor and Ricci tensor respectively. Let ∇_k be the operator of covariant differentiation with respect to Riemannian connection.

Weyl's projective curvature tensor $W_{kji}{}^h$ is given by

$$W_{kji}{}^h=R_{kji}{}^h-\frac{1}{n-1}(\delta_k^h R_{ji}-\delta_j^h R_{ki}). \tag{2.1}$$

Applying ∇_l to (2.1), we have by the assumption of parallel Weyl's projective curvature tensor,

(2.2) $$\nabla_l R_{kjih}=\frac{1}{n-1}(g_{kh}\nabla_l R_{ji}-g_{jh}\nabla_l R_{ki}).$$

On the other hand, from $R_{kjih}=R_{ihkj}$, we have

(2.3) $$\nabla_l R_{kjih}=\nabla_l R_{ihkj}.$$

Substituting (2.1) into (2.3), we have

$$g_{kh}\nabla_l R_{ji}-g_{jh}\nabla_l R_{ki}=g_{ij}\nabla_l R_{hk}-g_{hj}\nabla_l R_{ik}$$

from which,

(2.4) $$g_{kh}\nabla_l R_{ji}=g_{ji}\nabla_l R_{hk}.$$

Transvecting (2.4) with g^{kh}, we have

(2.5) $$n\nabla_l R_{ji}=g_{ji}\nabla_l R.$$

Moreover if we contract (2.5) with g^{lj}, then we have

$$\frac{(n-2)}{2}\nabla_i R=0$$

because, in general, we have

$$\frac{1}{2}\nabla_i R=\nabla^j R_{ji}$$

in any Riemannian space. Therefore we have

(2.6) $$\nabla_i R=0.$$

Taking account of (2.5) and (2.6), we have

$$\nabla_l R_{ji}=0.$$

Thus we have from (2.1),

THEOREM 2.1. *An n-dimensional* ($n>2$) *Riemannian space with parallel Weyl's projective curvature is locally symmetric.*

Next from (2.1), we have

$$S_{ml,kjih}=(\nabla_m\nabla_l R_{kjih}-\nabla_l\nabla_m R_{kjih})-\frac{1}{n-1}\{g_{kh}(\nabla_m\nabla_l R_{ji}-\nabla_l\nabla_m R_{ji})$$

$$-g_{jh}(\nabla_m\nabla_l R_{ki}-\nabla_l\nabla_m R_{ki})\}$$

where $S_{ml,kjih}=\nabla_m\nabla_l W_{kjih}-\nabla_l\nabla_m W_{kjih}$.

If we contract (2.1) by g^{mh}, then we have

(2.7) $$S_{ml,kji}{}^{m}=H_{ml,kji}{}^{m}-\frac{1}{n-1}\{\nabla_k\nabla_l R_{ji}-\nabla_l\nabla_k R_{ji})-(\nabla_j\nabla_l R_{ki}-\nabla_l\nabla_j R_{ki})\}$$

where $H_{ml,kji}{}^{m}=\nabla_m\nabla_l R_{kji}{}^{m}-\nabla_l\nabla_m R_{kji}{}^{m}$.

Multiplying (2.7) by $R^{kjil}=g^{ka}g^{jb}g^{ic}R_{abc}{}^{l}$, we have

$$R^{kjil}S_{ml,kji}{}^{m}=R^{kjil}H_{ml,kji}{}^{m}-\frac{1}{n-1}R^{kjil}(\nabla_k\nabla_l R_{ji}-\nabla_l\nabla_k R_{ji})-R^{kjil}(\nabla_j\nabla_l R_{ki}$$
$$-\nabla_l\nabla_j R_{ki})$$
$$=R^{kjil}H_{ml,kji}{}^{m}-\frac{2}{n-1}R^{kjil}(\nabla_k\nabla_l R_{ji}-\nabla_l\nabla_k R_{ji})$$

by virture of $R^{kjil}(\nabla_k\nabla_l R_{ji}-\nabla_l\nabla_k R_{ji})=R^{kjil}(\nabla_j\nabla_l R_{ki}-\nabla_l\nabla_j R_{ki})$.

Since $R^{kjil}=R^{jkli}=R^{lijk}$, the above equation can be written as

$$S_{ml,kji}{}^{m}R^{kjil}=H_{ml,kji}{}^{m}R^{kjil}. \tag{2.8}$$

Now when M is compact, we shall consider about Lichnerowitz formula ([1]). Denoting the volume element of M by dV, we have the following formula

$$\int_M [(\nabla_k R_{ji}-\nabla_j R_{ki})^2-K]\,dV=\frac{1}{2}\int_M (\nabla_l R_{kjih})^2\,dV \tag{2.9}$$

where K is a scalar function defined on M such that

$$K=R^{kjil}H_{ml,kji}{}^{m}.$$

If $S_{ml,kji}{}^{m}R^{kjil}\geq 0$ and $\nabla_k R_{ji}=\nabla_j R_{ki}$, then we have $\nabla_l R_{kji}{}^{h}=0$ taking account of (2.8) and (2.9).

Thus we have

THEOREM 2.2. *A compact n-dimensional Riemannian space with* $S_{ml,kji}{}^{m}R^{kjil}\geq 0$ *and* $\nabla_k R_{ji}=\nabla_j R_{ki}$ *is locally symmetric.*

§ 3. A Kählerian space with parallel H-projective curvature tensor.

An n-dimensional ($n\geq 2$) Kählerian space is a Riemannian space which admits a structure tensor $F_k{}^{j}$ satisfying

$$F_k{}^{a}F_a{}^{i}=-\delta_k^i,\quad \nabla_k F_j{}^{i}=0. \tag{3.1}$$

It is well known that the tensor H_{kj} defined by $H_{kj}=F_k{}^{a}R_{aj}$ satisfies

$$H_{kj}=-H_{jk}=-\frac{1}{2}F^{ba}R_{kjba}, \tag{3.2}$$
$$F_k{}^{a}H_{aj}=-H_{ka}F_j{}^{a}=-R_{kj}^{a}.$$

From the Bianchi's identity

$$\nabla_l R_{kjih}+\nabla_k R_{jlih}+\nabla_j R_{lkih}=0,$$

we deduce

$$\nabla_l H_{kj}+\nabla_k H_{jl}+\nabla_j H_{lk}=0 \tag{3.3}$$

by virtue of (3.1) and (3.2)([3]).

Transvecting (3.3) with $F_i^{\ l}$, we have

$$F_i^{\ l}\nabla_l H_{kj}=\nabla_j R_{ki}-\nabla_k R_{ji}. \tag{3.4}$$

Moreover, transvecting (3.4) with $F_b^{\ i}F_a^{\ k}$, we have

$$\nabla_b R_{aj}=F_b^{\ i}F_a^{\ k}(\nabla_j R_{ki}-\nabla_k R_{ji}). \tag{3.5}$$

H-projective curvature tensor $P_{kji}{}^h$ is given by

$$P_{kji}{}^h=R_{kji}{}^h+\frac{1}{n+2}(R_{ki}\delta_j^h-R_{ji}\delta_k^h+H_{ki}F_j^{\ h}-H_{ji}F_k^{\ h}+2H_{kj}F_i^{\ h}). \tag{3.6}$$

Applying ∇_h to (3.6), we obtain

$$\begin{aligned}\nabla_h P_{kji}{}^h&=\nabla_k R_{ji}-\nabla_j R_{ki}+\frac{1}{n+2}(\nabla_j R_{ki}-\nabla_k R_{ji}\\&\quad+F_j^{\ l}\nabla_l H_{ki}-F_k^{\ l}\nabla_l H_{ji}+2F_i^{\ l}\nabla_l H_{kj})\\&=\frac{n-2}{n+2}(\nabla_k R_{ji}-\nabla_j R_{ki})\end{aligned} \tag{3.7}$$

by virtue of (3.4).

If $\nabla_l P_{kji}{}^h=0$, then we have from (3.5) and (3.7)

$$\nabla_k R_{ji}=0. \tag{3.8}$$

Now applying ∇_l to (3.6), we have from (3.8)

THEOREM 3.1. *An n-dimensional ($n>2$) Kählerian space with parallel H-projective curvature tensor is locally symmetric.*

Otani College of Technology of Toyama and
Toyama University, Japan.

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