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EXTREMAL PROPERTIES OF p-FORMS ON A RIEMANNIAN MANIFOLD

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1. In this note we deduce extremal properties of p-forms φ which satisfy the differential equation

(1) $$\delta dS\varphi + d\delta T\varphi + PU\varphi = 0$$

on a Riemannian space, where S, T, U are appropriately chosen operators and P a smooth positive function. Work in this vein has recently been done by Kawai-Sario [6], where the extremal properties of harmonic ($\Delta\varphi=0$), semiharmonic ($\delta d\varphi=0$), cosemiharmonic ($d\delta\varphi=0$), quasiharmonic ($d\Delta\varphi=0$), and coquasiharmonic ($\delta\Delta\varphi=0$) forms has been systematically developed. Using their basic approach we consider, P-harmonic forms (cf. Duff [2,3]), biharmonic and k-harmonic forms.

2. Let M be a C^{∞}-Riemannian manifold of dimension n. We denote by $E^p(M)$ the vector space of smooth p-forms on M, d the exterior differential operator, and $*: E^p(M) \to E^{n-p}(M)$ the Hodge star operator. Then the codifferential operator δ is defined by

$$\delta\varphi = (-1)^{np+n+1} * d * \varphi, \quad \varphi \in E^p(M).$$

Also we have the relationship

$$**\varphi = (-1)^{np+p}, \quad \varphi \in E^p(M).$$

Let Ω be a regular subregion of M whose boundary is a C^{∞}-hypersurface $\partial\Omega$. An inner product on p-forms can be defined on Ω by

$$(\alpha, \beta) = \int_\Omega \alpha \wedge * \beta = \int_\Omega \alpha_{(i_1 \cdots i_p)} \beta^{(i_1 \cdots i_p)} * 1,$$

and the associated norm is given by $\|\alpha\|^2 = (\alpha, \alpha)$. Then Green's formula, which is of central importance, can be written

(2) $$(d\varphi, \phi) - (\varphi, \delta\phi) = \int_{\partial\Omega} \varphi \wedge * \phi,$$

where φ is a smooth $(p-1)$ form, and ϕ a smooth p-form.

We write $t\varphi$ and $n\varphi$ for the tangential and normal components of φ on $\partial\Omega$ respectively.

3. In the sequel we shall restrict our attention to the relatively compact subregion Ω, on which is defined a C^{∞}-function $P>0$, and assume that all forms are sufficiently smooth on $\partial\Omega$. We constru t a generalized energy integral

$$(3) \qquad [\varphi,\phi]=(dS\varphi,dS\phi)+(\delta T\varphi,\delta T\phi)+(PU\varphi,U\phi),$$

and corresponding norm $|\|\varphi\||^2=[\varphi,\varphi]$. The essence of our argument lies in the method of orthogonal projection given in the following form.

PROPOSITION 1. *If φ is a p-form, then among all p-forms ϕ with $\eta=\phi-\varphi$ such that $[\eta,\varphi]=0$, φ minimizes the functional*

$$|\|\phi\||^2=(dS\phi,dS\phi)+(\delta T\phi,\delta T\phi)+(PU\phi,U\phi),$$

that is,

$$|\|\phi\||^2=|\|\varphi\||^2+|\|\eta\||^2$$

Developing the inner product in (3) with $\eta=\phi-\varphi$, we obtain by Green's formula

$$\begin{aligned}[\eta,\varphi]&=(dS\eta,\ dS\varphi)+(\delta T\eta,\ \delta T\varphi)+(PU\eta,\ U\varphi)\\&=(S\eta,\ \delta dS\varphi)+\int_{\partial\Omega}S\eta\wedge{}^*dS\varphi+(d\delta T\varphi,\ T\eta)\\&\qquad-\int_{\partial\Omega}\delta T\varphi\wedge{}^*T\eta+(PU\eta,\ U\varphi).\end{aligned}$$

Therefore

$$(4) \qquad \begin{aligned}[\eta,\varphi]&=(S\eta,\delta dS\varphi)+(T\eta,d\delta T\varphi)+(U\eta,PU\varphi)\\&\qquad+\int_{\partial\Omega}S\eta\wedge{}^*dS\varphi-\delta T\varphi\wedge{}^*T\eta.\end{aligned}$$

The approach taken here, utilizing the operators S, T, and U, being more general than that in [6], permits us with the aid of Proposition 1 and (4) to deduce directly all the extremal properties which have been developed there. As an illustrative example, take $S=I$, $T=0$, and $U=0$. Then equation (1) becomes $\delta d\varphi=0$, i.e. φ is a semiharmonic form. Moreover (4) reads

$$\begin{aligned}[\eta,\varphi]&=(\eta,\delta d\varphi)+\int_{\partial\Omega}\eta\wedge{}^*d\varphi\\&=\int_{\partial\Omega}\eta\wedge{}^*d\varphi.\end{aligned}$$

From Proposition 1 we maintain (Kawai-Sario [6]):

THEOREM 1. *If φ is a semiharmonic form, then among those ϕ such that $t\phi=t\varphi$ on $\partial\Omega$, φ minimizes $(d\phi,d\phi)=\|d\phi\|^2$.*

Since a coclosed harmonic form φ ($\Delta\varphi=0$ and $\delta\varphi=0$) is clearly semiharmonic, and Duff [4] has established the existence of a coclosed harmonic form φ having preassigned boundary values of $t\varphi$, we obtain (Kawai-Sario [6]):

COROLLARY 1. *Among all p-forms ϕ with given boundary values $t\phi$, there is a form φ which minimizes $\|d\phi\|$, and φ is a coclosed harmonic form.*

Next, let $S=T=U=I$. Then (1) becomes

$$\Delta\varphi+P\varphi=0, \tag{5}$$

where $\Delta=\delta d+d\delta$ is the Laplace-Beltrami operator. A solution of (5) is called a *P-harmonic form*. In view of (5),

$$[\eta,\varphi]=(\eta,\delta d\varphi)+(\eta,d\delta\varphi)+(\eta,P\varphi)+\int_{\partial\Omega}\eta\wedge{}^*d\varphi-\delta\varphi\wedge{}^*\eta$$
$$=\int_{\partial\Omega}\eta\wedge{}^*d\varphi-\delta\varphi\wedge{}^*\eta.$$

THEOREM 2. *If φ is a P-harmonic form, then among forms ϕ such that $t\varphi=t\phi$, $n\varphi=n\phi$ on $\partial\Omega$, φ minimizes the energy integral for forms,*

$$|\|\phi\||^2=E(\phi)=(d\phi,d\phi)+(\delta\phi,\delta\phi)+(P\phi,\phi).$$

Explicitly,

$$E(\phi)=E(\varphi)+E(\eta) \quad \textit{where} \quad \eta=\phi-\varphi.$$

For functions, we refer to Kwon-Sario-Schiff [7, 8].

REMARK. The existence and uniqueness of a p-form φ satisfying (5) and with given boundary values of $\varphi=t\varphi+n\varphi$ was established by Duff [2].

4. Denote by $\mathscr{E}$ the space of p-forms with finite energy integral, and by $\mathscr{E}_P$ the subspace of P-harmonic forms, with $\mathscr{E}_0$ the subspace of forms φ such that $t\varphi=n\varphi=0$. In view of Theorem 2 and the Remark, the following orthogonal decomposition obtains:

$$\mathscr{E}=\mathscr{E}_P\oplus\mathscr{E}_0.$$

5. We next choose $S=T=\Delta=\delta d+d\delta$, $U=0$. Then (1) takes the form

$$\Delta^2\varphi=0, \tag{6}$$

that is, φ is a *biharmonic form.*

THEOREM 3. *If φ is a biharmonic form, then among forms ϕ such that $t\Delta\phi=t\Delta\varphi$, $n\Delta\phi=n\Delta\varphi$ on $\partial\Omega$, φ minimizes the Dirichlet integral for forms,*

$$D(\Delta\phi)=(d\Delta\phi,d\Delta\phi)+(\delta\Delta\phi,\delta\Delta\phi).$$

PROOF. Rewriting (4) we have

$$[\eta,\varphi]=(\Delta\eta,\delta d\Delta\varphi)+(\Delta\eta,d\delta\Delta\varphi)$$

$$+\int_{\partial\Omega}\Delta\eta\wedge{}^*d\Delta\varphi-\delta\Delta\varphi\wedge{}^*\Delta\eta$$

$$=\int_{\partial\Omega}\Delta\eta\wedge{}^*d\Delta\varphi-\delta\Delta\varphi\wedge{}^*\Delta\eta,$$

and the theorem follows. For functions, see Garabedian [5].

Finally, consider $S=T=\Delta^{k-1}$, $k=1, 2, 3, \cdots$, $U=0$. Then (1) is now

$$\Delta^k\varphi=0,$$

i.e. φ is a *k-harmonic form.*

THEOREM 4. *If φ is a k-harmonic form, the among forms ϕ with $t\Delta^{k-1}\phi=t\Delta^{k-1}\varphi$, $n\Delta^{k-1}\phi=n\Delta^{k-1}\varphi$ on $\partial\Omega$, φ minimizes the Dirichlet integral*

$$D(\Delta^{k-1}\phi)=(d\Delta^{k-1}\phi, d\Delta^{k-1}\phi)+(\delta\Delta^{k-1}\phi, \delta\Delta^{k-1}\phi).$$

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