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A NOTE ON THE GEOMETRIC MEANS OF ENTIRE FUNCTIONS OF TWO COMPLEX VARIABLES

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1. Let

$$f(z_1, z_2)=\sum_{k_1, k_2=0}^{\infty} a_{k_1 k_2} z_1^{k_1} z_2^{k_2},$$

be an entire function of two complex variables z_1 and z_2, holomorphic for $|z_t|\leq r_t$, $t=1,2$. We know that the maximum modulus of $f(z_1, z_2)$ for $|z_t|\leq r_t$ $(t=1,2)$ is denoted as

$$M(r_1, r_2)=\max_{|z_t|\leq r_t} |f(z_1, z_2)|, \ t=1,2.$$

The finite order ρ of an entire function $f(z_1, z_2)$ is denoted as ([1], p.219)

$$\lim_{r_1, r_2\to\infty} \sup \frac{\log\log M(r_1, r_2)}{\log(r_1 r_2)}=\rho.$$

The geometric means $G(r_1, r_2)$ and $g_k(r_1, r_2)$ of the function $|f(z_1, z_2)|$ for $|z_t|\leq r_t$ $(t=1,2)$ have been defined as ([2])

$$(1.1)\qquad G(r_1, r_2)=\exp\left\{\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \log|f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})|\,d\theta_1 d\theta_2\right\}$$

and

$$(1.2)\qquad g_k(r_1, r_2)=\exp\left\{\frac{(k+1)^2}{(r_1 r_2)^{k+1}}\int_0^{r_1}\int_0^{r_2} (x_1 x_2)^k \log G(x_1, x_2)\,dx_1 dx_2\right\},$$

where $0<k<\infty$.

In this note I have investigated a few properties of the above defined geometric means.

2. Let $\varphi(r_1, r_2)$ be a "slowly changing" function; that is $\varphi(r_1, r_2)>0$ and continuous for $r_1>r_1^0$, $r_2>r_2^0$ and for every constants $m, n>0$, $\varphi(mr_1, nr_2)\sim\varphi(r_1, r_2)$ as r_1 or r_2 or r_1 and r_2 tend to infinity.

Also let us set

$$(2.1)\qquad \lim_{r_1, r_2\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log G(r_1, r_2)}{(r_1 r_2)^{\rho}\varphi(r_1, r_2)}=\begin{matrix}c\\ d\end{matrix} \quad (0<d\leq c<\infty)$$

and

$$\lim_{r_1, r_2 \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log g_k(r_1, r_2)}{(r_1 r_2)^\rho \varphi(r_1, r_2)} = \begin{matrix} p \\ q \end{matrix} \quad (0 < q \leq p < \infty). \tag{2.2}$$

In my earlier paper ([2]), I have proved the following result:
If $f(z_1, z_2)$ be an entire function of finite nonzero order ρ, then

$$\left\{ \frac{k+1}{k+\rho+1} \right\}^2 d \leq q \leq p \leq \left\{ \frac{k+1}{k+\rho+1} \right\}^2 c. \tag{2.3}$$

Now I intend to prove the following theorems:

THEOREM 1. *Let $f(z_1, z_2)$ be an entire function of order ρ, then*

$$\left\{ \frac{k+1}{k+\rho+1} \right\}^2 \frac{d}{c} \leq \lim_{r_1, r_2 \to \infty} \frac{\log g_k(r_1, r_2)}{\log G(r_1, r_2)} \leq \left\{ \frac{k+1}{k+\rho+1} \right\}^2 \frac{c}{d}.$$

PROOF. From (2.1) and (2.2), we obtain

$$\frac{q-\varepsilon}{c+\varepsilon} < \frac{\log g_k(r_1, r_2)}{\log G(r_1, r_2)} < \frac{p+\varepsilon}{d-\varepsilon}.$$

Taking limits and using (2, 3), the result follows.

COROLLARY. *If $c=d$, then*

$$(k+1)^2 \log G(r_1, r_2) \sim (k+\rho+1)^2 \log g_k(r_1, r_2).$$

THEOREM 2. *Let $f(z_1, z_2)$ be an entire function and if $0 < r_1 < R_1$, $0 < r_2 < R_2$, then*

$$\begin{aligned} \{(R_1R_2)^{k+1} - (r_1r_2)^{k+1}\} \log G(r_1, r_2) &\leq \{(R_1R_2)^{k+1} \log g_k(R_1, R_2) \\ &- (r_1r_2)^{k+1} \log g_k(r_1, r_2)\} \leq \{(R_1R_2)^{k+1} - (r_1r_2)^{k+1}\} \log G(R_1, R_2) \end{aligned} \tag{2.4}$$

PROOF. Since $G(r_1, r_2)$ is an increasing function of r_1 and r_2, therefore from (1.2) we have

$$\begin{aligned} &(R_1R_2)^{k+1} \log g_k(R_1, R_2) - (r_1r_2)^{k+1} \log g_k(r_1, r_2) \\ &= (k+1)^2 \left\{ \int_0^{R_1} \int_0^{R_2} - \int_0^{r_1} \int_0^{r_2} \right\} (x_1x_2)^k \log G(x_1, x_2) dx_1 dx_2 \\ &\leq \{(R_1R_2)^{k+1} - (r_1r_2)^{k+1}\} \log G(R_1, R_2). \end{aligned}$$

Also

$$\begin{aligned} &(R_1R_2)^{k+1} \log g_k(R_1, R_2) - (r_1r_2)^{k+1} \log g_k(r_1, r_2) \\ &= (k+1)^2 \left\{ \int_0^{R_1} \int_0^{R_2} - \int_0^{r_1} \int_0^{r_2} \right\} (x_1x_2)^k \log G(x_1, x_2) dx_1 dx_2 \end{aligned}$$

$$\geq \{(R_1R_2)^{k+1} - (r_1r_2)^{k+1}\} \log G(r_1, r_2).$$

Hence the result follows.

COROLLARY. *if* η $(0<\eta<1)$ *is a constant, then*

$$\lim_{r_1, r_2 \to \infty} \left[\frac{\{g_k(\beta r_1, \beta r_2)\}^{\beta^{2(k+1)}}}{g_k(r_1, r_2)} \right] = 0.$$

Putting $r_1 = \beta r_1$, $r_2 = \beta r_2$ and $R_1 = r_1$, $R_2 = r_2$ in (2.4) and taking the limit the result follows.

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REFERENCES

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