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ON NORMS OF MULTILINEAR SYMMETRIC OPERATORS

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Let E and F be Banach spaces and let $L_s^r(E,F)$ denote the Banach space of continuous r-linear symmetric operators with the operator norm

$$|\ \ |: L_s^r(E,\ F) \to R$$

$$|A| = \sup_{|z_i|\leq 1} |A(z_1, \cdots, z_r)| \text{ for } A \in L_s^r(E,F)$$

Define a function $|\ \ |_1 : L_s^r(E,F) \to R$ by

$$|A|_1 = \sup_{|z|\leq 1} |Az^r| \text{ for } A \in L_s^r(E,F), \quad \text{where } z^r \text{ denotes the } r\text{-tuple } (z, \cdots, z).$$

It is clear that the norm properties are satisfied:

$$|A|_1 \geq 0,$$

$$|A+B|_1 \leq |A|_1 + |B|_1$$

and $|\lambda A|_1 = |\lambda|\,|A|_1$ for $\lambda \in R$, $A,\ B \in L_s^r(E,F)$.

To see that $|A|_1 = 0$ implies $A=0$, we define the map $\varphi: E \to F$ by $\varphi(h) = Ah^r = 0$, and consider $A = \overline{A}(0)$ where $\overline{A}: E \to L_s^r(E,\ F)$ given by $\overline{A}(z) = A$ for $z \in E$. Then by the converse of Taylor's theorem [1] [3], it follows $\overline{A} = D^r\varphi \equiv 0$. Hence $A=0$.

We note that A is completely determined by its values on the diagonal. For if $Az^k = Bz^k$ and $A,\ B \in L_s^r(E,\ F)$ for every $z \in E$, then $|A-B|_1 = 0$ and consequently $A-B=0$, i.e. $A=B$.

We wish to show that these two norms are equivalent. To prove this, we need the following theorem [2] which can also be used to prove that $|A|_1 = 0$ implies $A=0$.

THEOREM. *Let γ be a non-negative real number. Let E and F be Banach spaces and U an open subset of E. Let $f: U \to L_s^k(E,\ F)$. If there exists constants $a,\ b > 0$ such that $f(y)z^k = 0(z^{k+\gamma})$ for $a|y| < |z| < b|y|$ and $z, y \in E$, then $f(y) = 0(y^\gamma)$.*

THEOREM. *The norms* $|\ |$ *and* $|\ |_1$ *are equivalent.*

PROOF. Clearly, $|A|_1 \leq |A|$. Therefore, it suffices to show a sequence $\{A_n\}$ in $L_s^r(E, F)$ converging to 0 relative to $|\ |_1$, also converges to 0 relative to $|\ |$. Define a function

$$f : U \to L_s^r(E, F) \text{ by}$$

$$f(x) = A_{\left[\frac{1}{|x|}\right]} \quad \text{for } x \neq 0,$$

$$\text{and } f(0) = 0$$

where U is an open disk of radius 1, and $\left[\frac{1}{|x|}\right]$ denotes the largest integer contained in $\frac{1}{|x|}$. Choose any a, b with $b > a > 0$, then for $a|x| < |z| < b|x|$

$$|f(x)z^r| = |A_n z^r| = \left| A_n\left(\frac{z}{|z|}\right)^r \right| |z|^r \text{ where } n = \left[\frac{1}{|x|}\right].$$

Noting that $\left| A_n\left(\frac{z}{|z|}\right)^r \right| \leq |A_n|_1, n \to \infty$ as $z \to 0$, and $|A_n|_1 \to 0$, we have $f(x)z^r = 0(z^r)$.

By the preceeding theorem, we see immediately that $f(x) = 0(1)$, that is $f(x) \to 0$ as $x \to 0$ relative to the standard norm $|\ |$. Hence $f(\frac{1}{n}) = A_n \to 0$ as $n \to \infty$ relative to $|\ |$.

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