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ON MAXIMAL GOLDIE SUBRINGS OF ABSOLUTELY SEMIPRIME GOLDIE RINGS

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A maximal Goldie subring of a ring is defined to be a maximal element in the set of all right Goldie subrings properly contained in the ring. Let T be a semiprime right Goldie ring which is not prime. In this paper, we study a certain subset of the set of all maximal Goldie subrings of T. We partition this subset into equivalence classes and consider properties which members of a particular class have in common. Some maximal Goldie subrings are semiprime right Goldie rings. Others are subdirect sums of rings with infinite direct sums of nonzero right ideals or direct summands of T.

Throughout this paper, ***Goldie ring*** will denote ***right Goldie ring***.

In [2], Goldie characterized the classical quotient ring of a semiprime Goldie ring. Later, Small [4] proved the following proposition: If R is a semiprime subring of the complete $n\times n$ matrix ring over C where C is a commutative semiprime Goldie ring, then R is a right and left Goldie ring. Before stating the related question which we will investigate, we mention some remarks concerning the existence of maximal Goldie subrings.

Certainly a proper maximal subring of a ring only needs to be Goldie in order to be a maximal Goldie subring. So maximal Goldie subrings do exist in some rings. But every Goldie subring is not necessarily contained in a maximal Goldie subring. For example, let R be the complete direct sum of a denumerably infinite number of copies of Z_2 where Z_2 is the ring of integers modulo 2. The ring R has no maximal Goldie subrings.

DEFINITION. A ring is *absolutely semiprime* if it is semiprime but not prime.

Now let T be a ring with ascending chain condition on right annihilators such that T is not prime. Let K be the set of maximal right annihilators of nonzero right ideals of T. We note that $K=\{r(A_i):i\in I\}$ where A_i is the left annihilator of an element of $K, r(A_i)$ is the right annihilator of A_i, and I is an indexing set.

Suppose that S is a subring of the ring T. Each of the $r(A_i)$ satisfies one of the following: 1) $r(A_i)\subset S$, 2) $S+r(A_i)=T$ where $S+r(A_i)$ is the subring of T generated by S and $r(A_i)$, 3) $S\subsetneqq S+r(A_i)\subsetneqq T$. With reference to the above, we shall at times say that a maximal right annihilator of a nonzero right ideal is of Type 1, 2 or 3 with respect to S. Let $\mathscr{S}$ be the set of all maximal Goldie subrings S of the ring T which possess the property that each member of K is of the same Type with respect to S. Then we can partition $\mathscr{S}$ into four disjoint subsets:

Subset 1 $=\{S\in\mathscr{S} : A_i\cap S=0$ for some $i\in I\}$

Subset 2 $=\{S\in\mathscr{S} : A_i\cap S\neq 0$ and $r(A_i)\subset S$ for all $i\in I\}$

Subset 3 $=\{S\in\mathscr{S} : A_i\cap S\neq 0$ and $S+r(A_i)=T$ for all $i\in I\}$

Subset 4 $=\{S\in\mathscr{S} : A_i\cap S\neq 0$ and $S\subsetneqq S+r(A_i)\subsetneqq T$ for all $i\in I\}$.

In this paper, we prove the following result: If T is absolutely semiprime Goldie, then each member of Subset 1 is a direct summand of T, each member of Subset 2 or Subset 3 is semiprime Goldie, and each member of Subset 4 is a subdirect sum of rings with infinite direct sums of nonzero right ideals.

Herstein proved that a Goldie ring has only a finite number of maximal right annihilators of nonzero right ideals [3]. The following proposition provides the same conclusion but has different hypothesis.

PROPOSITION 1. *Let T be a semiprime ring with a.c.c. on right annihilators. Then K has only a finite number of distinct elements.*

PROOF. Suppose K is infinite. Then by a Lemma in [3], a countable number of the A_i will form an infinite direct sum, $A_1+\cdots+A_n+\cdots$. Now $r(A_k+A_{k+1}+\cdots)$, $k=1,2,\cdots$ form an ascending chain of right annihilators. Since T has a.c.c. on right annihilators, there exists a positive integer N such that $r(A_N+\cdots)=r(A_{N+j}+\cdots)$, $j=1,2,\cdots$. However, A_N annihilates A_{N+j}, $j=1,2,\cdots$ [3]. Therefore, $(A_N)^2=0$. But this contradicts the semiprimeness of T. Hence K is finite.

Since I is a finite set, we let $I=\{1,2,\cdots,n\}$ where n is a positive integer.

The following results which can be found in [3] or are immediate consequences of lemmas in [3] are included here for easy reference.

PROPOSITION 2. *Let T be a ring with a.c.c. on right annihilators. Furthermore, suppose that T is not prime.*

a) If $r(A_i)$ is an element of K, then $r(A_i)$ is a prime ideal.

b) If $r(A_i)\neq r(A_j)$, then $A_i\subset r(A_j)$.

c) $A_1+\cdots+A_n$ is a direct sum.

d) If T is absolutely semiprime, then $A_i \cap r(A_i)=0$.

e) T is absolutely semiprime if and only if the intersection of the elements of K is the zero ideal.

f) If T is absolutely semiprime, then T has at least two maximal right annihilators of nonzero right ideals.

g) A subring inherits a.c.c. on right annihilators from a ring.

h) In any semiprime ring, if U and V are ideals such that $UV=0$, then $VU=0$.

1. Quotients Rings.

In this section, we prove that if S is a member of Subset 2 or Subset 3, then S is semiprime Goldie. Furthermore, we are able to show that a member of Subset 3 has a quotient ring which is isomorphic to the quotient ring of T.

Throughout the remainder of the paper, the terminology of the preceding discussion will be employed in a manner consistent with its previous description.

THEOREM 3. *Let S be a Goldie subring of an absolutely semiprime ring T with a.c.c. on right annihilators, and suppose each element of K is contained in S. Then S is semiprime, and S has a quotient ring which is semisimple with descending chain condition on right ideals.*

PROOF. By Proposition 2*f*, K has at least two distinct elements, $r(A_1)$ and $r(A_2)$. Suppose that N is a nilpotent ideal of S. Then $Nr(A_1)$ and $Nr(A_2)$ are nilpotent right ideals of T, and so $Nr(A_1)=0$ and $Nr(A_2)=0$. Since A_i is the left annihilator of $r(A_i)$ then $N \subset A_1 \cap A_2$. By Proposition 2*c*, it follows that $N=0$. We apply Goldie's Theorem [2] to obtain the conclusion of Theorem 3.

PROPOSITION 4. *Let S and T be as in Theorem 3. Then the set of maximal right annihilators of nonzero right ideals of T coincides with the set of maximal right annihilators of nonzero right ideals of S.*

PROOF. Suppose $r(A_i)$ is an element of K. By Proposition 2*b*, A_i is contained in S. Hence $r(A_i) \subset r_S(B)$ where $r_S(B)$ is a maximal right annihilator of a nonzero right ideal of S. Therefore, $Br(A_i)=0$, and so $B \subset A_i$. By Proposition 2*h* and Proposition 2*d*, BA_i is a nonzero right ideal of T which is contained in B. Now $r(A_i) \subset r_S(B) \subset r(BA_i)$. However, the maximality of $r(A_i)$ implies that $r(BA_i)=r(A_i)$. Therefore, $r(A_i)=r_S(B)$ and $r(A_i)$ is a maximal right annihilator of a nonzero right ideal of S.

Suppose $r_S(B)$ is a maximal right annihilator of a nonzero right ideal in S. By Proposition 2e, B is not contained in $r(A_j)$ for some j and $A_jB\neq 0$. By Proposition 2h, $BA_j\neq 0$. Since $A_j\subset S$, then $BA_j\subset B$ and $r_S(B)\subset r_S(BA_j)\subset r_S(A_j)$. The maximality of $r_S(B)$ implies that $r_S(B)=r_S(A_j)=r(A_j)$. Therefore, $r_S(B)$ is a maximal right annihilator of a nonzero right ideal of T.

We now consider the members of Subset 3.

THEOREM 5. *Let S be a Goldie subring of an absolutely semiprime Goldie ring T. Suppose S satisfies the conditions imposed on members of Subset 3. Then S is semiprime Goldie and S and T have the same quotient ring.*

PROOF. Since $S+r(A_i)=T$, $i=1,2,\cdots,n$, $A_i\cap S$ is an ideal of T. By the maximality of $r(A_i)$, $r(A_i)\cap S=r_S(A_i\cap S)$. Using a proof similar to that of [3, Lemma 1], we can show that $r_S(A_i\cap S)$ is a prime ideal of S. Suppose that $UV\subset r_S(A_i\cap S)$ where U and V are ideals of S. Then $(A_i\cap S)UV=0$. If $(A_i\cap S)U=0$, then we are finished. So suppose $(A_i\cap S)U\neq 0$. Thus $V\subset r_S[(A_i\cap S)U]$. However, $(A_i\cap S)U$ is a right ideal of T since $S+r(A_i)=T$. Therefore, $r[(A_i\cap S)U]=r(A_i)$. Since $r_S(A_i\cap S)U]=r(A_i)\cap S$, then $V\subset r_S(A_i\cap S)$. We conclude that $r(A_i)\cap S$ is a prime ideal and $S/r(A_i)\cap S$ is a prime ring.

Suppose N is a nilpotent ideal of S. Then N is contained in $r(A_i)\cap S$, $i=1,2,\cdots,n$. By Proposition 2e, $N=0$. Therefore, S is semiprime.

In [3], Herstein proved that a semiprime Goldie ring T has $Q(T/r(A_i))\oplus\cdots\oplus Q(T/r(A_n))$ as its quotient ring where $Q(T/r(A_i))$ denotes the classical quotient ring of $T/r(A_i)$. Since $r(A_i)\cap S$ is a right annihilator of a right ideal, $S/r(A_i)\cap S$ has a.c.c. on right annihilators [3, Lemma 4]. Furthermore, by mimicking the the proofs of two results of Herstein, we can show quite easily that $S/r(A_i)\cap S$ has no infinite direct sum of nonzero right ideals [3, Lemma 4] and that $Q(S/r(A_1)\cap S)\oplus\cdots\oplus Q(S/r(A_n)\cap S)$ is the quotient ring for S [3, Teorema]. In order to show that $S/r(A_i)\cap S$ and $T/r(A_i)$ have the same quotient ring, it suffices to show that $S/r(A_i)\cap S$ and $T/r(A_i)$ are isomorphic. But since $S+r(A_i)=T$, this follows from the Isomorphism Theorem.

2. Direct Sums.

For the remainder of the article, IDS will denote ***infinite direct sum of nonzero right ideals.***

In this section, we show that T is the direct sum of S and A_i where S belongs to Subset 1 and $A_i \cap S=0$. First we need the following lemma.

LEMMA 6. *Let S and J be subrings of an absolutely semiprime Goldie ring T such that S has no IDS and $sJ \subset J$ and $Js \subset J$ for each s belonging to S. If $S+J$ has an IDS and $S/S\cap J$ has no IDS, then J has an IDS.*

PROOF. We will prove the contrapositive of the lemma. It suffices to show that if J has no IDS and $S+J$ has an IDS, then $S/S\cap J$ has an IDS. Suppose $I_1+\cdots+I_n+\cdots$ is an IDS in $S+J$. Since S and J have no IDS, we may assume that $I_i \cap S=0$ and $I_i \cap J=0$ for $i=1, 2, \cdots$. Let $S_i=\{s \in S : \exists ay \in J$ such that $s+y \in I_i\}$, $i=1, \cdots, n$. Then S_i is a right ideal in S. Also S_i is not contained in $S\cap J$. For if $x \in S\cap J$ and $y \in J$ such that $x+y \in I_i$, then $x+y \in I_i \cap J$. $I_i \cap J=0$ would imply that $x+y=0$. Since $I_i \neq 0$, then S_i is not contained in $S\cap J$.

We claim that there is an infinite sequence of S_i, say S_N, S_{N+1}, $\cdots$ where N is a positive integer, such that $S_j \cap \sum_{k=N}^{j-1} S_k \subset S\cap J$ for $j \geq N$. We assume that the claim is false. Then there exists a positive integer $N(1)$ such that $S_{N(1)} \cap \sum_{k=1}^{N(1)-1} S_k \not\subset S\cap J$. Hence there exists s_k belonging to S_k, $k=1,2, \cdots, N(1)$ such that $s_{N(1)}$ is not in $S\cap J$ and

$$s_1+s_2+\cdots+s_{N(1)-1}-s_{N(1)}=0. \tag{1}$$

Now there corresponds to each s_k some y_k in J such that $s_k+y_k \in I_k$. Let $t_k=s_k+y_k$ and $t=t_1+\cdots+t_{N(1)-1}-t_{N(1)}$. Thus t belongs to $J_1=I_1+\cdots+I_{N(1)}$. Now $s_{N(1)} \notin S\cap J$ implies that $t_{N(1)} \neq 0$. Since J_1 is a direct sum, it follows that t is nonzero. Using Equation (1), we find that $y_1+y_2+\cdots+y_{N(1)-1}-y_{N(1)}$ is a nonzero element of J_1. But then $J_1 \cap J$ is a nonzero right ideal of J. Now there exists an $N(2) > N(1)$ such that $S_{N(2)} \cap \sum_{k=N(1)+1}^{N(2)-1} S_k \not\subset S\cap J$. We let $J_2=I_{N(1)+1}+\cdots+I_{N(2)}$. It is easily seen that $J_2 \cap J$ is a nonzero right ideal of J. By induction, we can define $J_i, i=1,2, \cdots$ such that $J_1+\cdots+J_n+\cdots$ is an IDS in J. But, by assumption, J has no IDS. Hence the claim is proven. So we might as well assume that $N=1$.

Let S_i' be the image of S_i in $S/S\cap J$ under the natural homomorphism. We claim that there is a positive integer M such that $S_M'+S_{M+1}'+\cdots$ is an IDS in $S/S\cap J$. $S_j' \neq 0$, $j=1,2, \cdots$. since $S_j \not\subset S\cap J$. Now suppose that no such M exists. Then there is a positive integer $M(1)$ such that $S_{M(1)}' \cap \sum_{j=1}^{M(1)-1} S_j' \neq 0$. Hence there are $s_k' \in S_k'$ where $s_k \in S_k$, $k=1,2, \cdots, M(1)$ such that $s_{M(1)}' \neq 0$ and

(2) $$s_1'+\cdots+s_{M(1)-1}'-s_{M(1)}'=0.$$

But $s_{M(1)}'\neq 0$ implies that $s_{M(1)}\notin S\cap J$. So there corresponds to $s_{M(1)}$ some $y_{M(1)}$ in J such that $s_{M(1)}+y_{M(1)}$ is a nonzero element of $I_{M(1)}$. By Equation (2), $s_1+\cdots+s_{M(1)-1}-s_{M(1)}$ belongs to $S\cap J$. If $y_k\in J$ such that $r_k=s_k+y_k$ is an element of I_k, then $r=r_1+\cdots+r_{M(1)-1}-r_{M(1)}$ belongs to $L_1=I_1+\cdots+I_{M(1)}$. But since $r_{M(1)}\neq 0$, r is a nonzero element of $L_1\cap J$. Similarly, we choose $M(2)>M(1)$ such that $S_{M(2)}'\cap\sum_{j=M(1)+1}^{M(2)-1}S_j'\neq 0$. We can show that $L_2\cap J\neq 0$ in the same manner as we did L_1. Continuing by induction, we find that $(L_1\cap J)+\cdots+(L_n\cap J)+\cdots$ is an IDS in J. But we have contradicted the assumption that J has no IDS. Therefore, the claim has been proven. Hence $S/S\cap J$ has an IDS.

THEOREM 7. *Let S be a maximal Goldie subring of an absolutely semiprime Goldie ring T. If $A_i\cap S=0$ for some i, then T is the direct sum of the additive groupstructures of S and A_i.*

PROOF. Since $A_i\cap S=0$ and $A_i\neq 0$, then S is properly contained in $S+A_i$. Assume $S+A_i$ is properly contained in T. $S+A_i$ inherits a.c.c. on right annihilators from T. However the maximality of S implies that $S+A_i$ is not Goldie. Therefore $S+A_i$ has an IDS. We can apply Lemma 6 to show that A_i has an IDS, $I_1+\cdots+I_n+\cdots$. Then $I_1A_i+\cdots+I_nA_i+\cdots$ is a direct sum of right ideals in T. Therefore, there exists a positive integer N such that $J_NA_i+J_{N+1}A_i+\cdots=0$. By Proposition 2*h*, $J_N+J_{N+1}+\cdots$ is contained in $r(A_i)$. Thus $A_i\cap r(A_i)\neq 0$, and Proposition 2*d* has been contradicted. We conclude that $S+A_i=T$.

3. Subdirect Sums.

In this section, a necessary and sufficient condition for a maximal Goldie subring of T to belong to Subset 4 is given.

LEMMA 8. *Let S be a maximal Goldie subring of the absolutely semiprime Goldie ring T. There cannot exist simultaneously in T maximal right annihilators of Type 1 and Type 3 with respect to S.*

PROOF. If $r(A_1)$ is of Type 1 and $r(A_2)$ is of Type 3 with respect to S, then $S+r(A_2)$ has an IDS, $I_1+\cdots+I_n+\cdots$. It follows easily that there exists an N such that $I_N+\cdots\subset A_1\cap A_2$. However, we have contradicted Proposition 2*d*.

THEOREM 9. *Let S be a maximal Goldie subring of an absolutely semiprime Goldie ring T. Then S is a member of Subset 4 if and only if S is a subdirect sum of the* *rings* $S/r(A_i)\cap S, i=1,2,\cdots,n$ *and each* $S/r(A_i)\cap S,\ i=1,2,\cdots,n$ *has an IDS.*

PROOF. First we prove the necessary portion. Since, by Proposition 2*e*, $[r(A_1)\cap S]\cap\cdots\cap[r(A_n)\cap S]=0$, then S is a subdirect sum of the rings $S/r(A_i)\cap S$. Since $r(A_i)$ is of Type 3, $S+r(A_i)$ has an IDS. But $r(A_i)$ has no IDS. For if $r(A_i)$ had an *IDS*, $I_1+\cdots+I_n+\cdots$, then there would exist a positive integer N such that $I_N+I_{N+1}+\cdots\subset r(A_i)\cap A_i$. Applying the contrapositive of Lemma 6, we find that $S/r(A_i)\cap S,\ i=1,2,\cdots,n$ has an IDS.

Now we prove sufficiency. Suppose S is a subdirect sum of rings $S/r(A_i)\cap S$, $i=1,\cdots,n$ with IDS. Suppose that $r(A_j)$ is of Type 2 and $A_j\cap S\neq 0$. Even if S is not semiprime, we can use the semiprimeness of T to show, as we mentioned in the proof of Theorem 5, that $S/r(A_j)\cap S$ has no IDS. Now suppose $r(A_j)$ is of Type 2 and $A_j\cap S=0$. By Theorem 7, $S+A_j=T$. It follows from Proposition 2*b* that $S+r(A_i)=T,\ i\neq j$. Since $S/r(A_i)\cap S$ has IDS, then $A_i\cap S=0,\ i\neq j$. Hence $A_i\cap(r(A_j)\cap S)=0,\ i=1,\cdots,n$. Also, $r(A_j)\cap S$ is an ideal of T since $S+A_j=T$. Thus $A_i\subset r[r(A_j)\cap S],\ \ i=1,\cdots,n$. Furthermore, $r(A_j)\cap S\neq 0$ since otherwise $S/r\ (A_j)\cap S$ would be Goldie. Since $r[r(A_j)\cap S]$ is contained in a maximal right annihilator of a nonzero right ideal, there exists a positive integer k, $1\leq k\leq n$, such that $A_i\subsetneq r(A_k),\ i=1,\cdots,n$. But then $A_k\cap r(A_k)\neq 0$. Therefore, Proposition 2*d* is contradicted. We conclude that no element of K is of Type 2.

Now each $r(A_i)$ is of Type 1 or Type 3. But by Lemma 8, right annihilators of both Type 1 and Type 3 with respect to S cannot exist in T. We assume that the $r(A_i), i=1,\cdots,n$ are of Type 1. By Theorem 3, S is semiprime. By Theorem 4, each $r(A_i),\ i=1,\cdots,n$, is a maximal right annihilator of a nonzero right ideal of S. By [3, Lemma 4], $S/r(A_i)\cap S$ has no IDS. Therefore, $r(A_1),\cdots,r(A_n)$ are of Type 3. If $A_i\cap S=0$ for some i, then $S+A_i=T$. Thus $A_i\cap S\neq 0,\ i=1,\ \cdots,\ n$. Therefore, S belongs to Subset 4.

Now let us suppose that S is a maximal Goldie subring of T such that some elements of K are of one Type and some elements of K are of another Type with respect to S. Lemma 8 tells us that there are only two possibilities: elements of Type 1 and Type 2 or elements of Type 2 and Type 3. Results of the author's work in these cases are partial at present.

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