Kyungpook Math. J.
Volume 12, Number 1
June, 1972

THE STRUCTURE OF A CLASS OF REGULAR SEMIGROUPS

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We describe the structure of regular semigroups in which the idempotents form a semigroup (variously termed orthodox [3] or strictly regular semigroups [12]) modulo $\mathscr{L}$-unipotent semigroups (semigroups in which each $\mathscr{L}$-class ($\mathscr{L}$ is Green's relation) contains precisely one idempotent) and bands of rectangular bands (this gives a finer description than a description mod bands) (theorem 8). The structure of $\mathscr{L}$-unipotent semigroups was given mod inverse semigroups and semilattices of right zero semigroups in [10] and special classes of $\mathscr{L}$-unipotent semigroups have been structured more finely (see for example [7], [8], and [9]). In [11], we gave a structure theorem for generalized $\mathscr{L}$-unipotent semigroups (regular semigroups whose set of idempotents E satisfy the condition: $e, f \in E$ and $ef=e$ imply $gegfe=ge$ for all $g \in E$). A second structure theorem (mod $\mathscr{L}$-unipotent semigroups and bands of left zero semigroups) for these semigroups is given here as a corollary (corollary 9) to theorem 8. Yamada [12] gives another structure theorem for regular semigroups in which the idempotents form a semigroup.

Unless otherwise stated we use the definitions and notation of [1]. Let us first state the structure theorem for $\mathscr{L}$-unipotent semigroups. Let X be an inverse semigroup with semilattice of idempotents Y, and let E be a semilattice Y of right zero semigroups $\{E_y : y \in Y\}$. Let $r \to \alpha_r$ be a mapping of X into $\mathscr{T}_E$, the full transformation semigroup on E, subject to the conditions I(a) $E_y\alpha_r \subset E_r - 1_{yr}$ (b) *if* $g_x \in E_x$ and $h_y \in E_y$, $(g_x h_y)\alpha_r = (g_x\alpha_{yr})(h_y\alpha_r)$; II $\alpha_t\alpha_z\rho_e = \alpha_{tz}\rho_e$ for all $e \in E_{(tz)} - 1_{tz}$ where ρ_e is the inner right translation of E determined by e. Let (X, E, Y, α) denote $\{(s, g_{s^{-1}s}) : s \in X$ and $g_{s^{-1}s} \in E_{s^{-1}s}\}$ under the multiplication $(s, g_{s^{-1}s})(t, h_{t^{-1}t}) = (st, (g_{s^{-1}s}\,\alpha_t)h_{t^{-1}t})$.

THEOREM 1. (Warne, [10]). *S is an $\mathscr{L}$-unipotent semigroup if and only if $S \cong (X, E, Y, \alpha)$ for some collection X, E, Y, α.*

In lemmas 2—7, S will denote a regular semigroup whose set of idempotents form a semigroup. If $a \in S$, $\mathscr{I}(a)$ will denote the collection of inverses of a. A congruence ρ will be termed $\mathscr{L}$-unipotent if S/ρ is an $\mathscr{L}$-unipotent semigroup.

$\mathscr{R}$, $\mathscr{L}$ and $\mathscr{D}$ will denote Green's relations. If A is a semigroup, E_A will denote the set of idempotents of A.

LEMMA 2. *Let* $\rho=\{(a,b)\in S^2: a=su,\ b=sv$ *for some* $(u,v)\in\mathscr{L},\ \mathscr{I}(u)=\mathscr{I}(v),\ s\in S^1\}$. *Then,* ρ, *the transitive closure of* ρ, *is the smallest* $\mathscr{L}$*-unipotent congruence on* S.

PROOF. By [3, theorem 3], $\{(a,b)\in S^2: \mathscr{I}(a)=\mathscr{I}(b)\}$ is a congruence on S. Thus, it follows from [2, lemma 10.3] or directly that ρ^t is a left congruence on S, and, hence, clearly, ρ^t is a congruence on S. By [5 : 1, p. 129, Ex. 1], E_s is a semilattice Ω of rectangular bands $\{E_\alpha:\alpha\in\Omega\}$. Let $X_\alpha=E_\alpha\rho^t$ and let $x,y\in X_\alpha$ with $x=e\rho^t$ and $y=f\rho^t$ for some $e,f\in E_\alpha$. Since $efe=e$ and $fef=f$, $\mathscr{I}(e)=\mathscr{I}(f)$ ($e\in\mathscr{I}(e)\cap\mathscr{I}(f)$ implies $\mathscr{I}(e)=\mathscr{I}(f)$ by [3, theorem 2]) and, hence, $\mathscr{I}(ef)=\mathscr{I}(f)$. Thus, $(ef,f)\in\mathscr{L}$ implies $(ef,f)\in\rho^t$ and, hence, $xy=y$. Since $(e,f)\in\rho^t$ implies $\mathscr{I}(e)=\mathscr{I}(f)$, $X_\alpha\cap X_\beta=\phi$ if $\alpha\neq\beta$. Thus, since $E_S\rho^t=E_{S/\rho^t}$ ([4, lemma 2.2]), E_{S/ρ^t} is the semilattice Ω of right zero semigroups $\{x_\alpha:\alpha\in\Omega\}$. Hence, ρ^t is an $\mathscr{L}$-unipotent congruence by [10, proposition 5]. Finally, let δ be an $\mathscr{L}$-unipotent congruence on S. If $(u,v)\in\mathscr{L}$ and $v'\in\mathscr{I}(v)=\mathscr{I}(u)$, then $uv'\in E_S$, $(uv',vv')\in\mathscr{L}$ and, hence, $(uv',vv')\in\delta$ or $(u,v)\in\delta$. Thus, $\rho^t\subset\delta$.

For brevity, we let $\lambda=\rho^t$ and $X=S/\lambda$. Thus, if $\lambda_s=s\lambda^{-1}$ for $s\in X$, $\{\lambda_s:s\in X\}$ is the collection of λ-classes of S and $\lambda_s\lambda_t\subset\lambda_{st}$. If $s\in E_X$, let $E_s=\lambda_s$.

LEMMA 3. E_S *is the band* E_X *of rectangular bands* $\{E_s:s\in E_X\}$.

PROOF. By [4, lemma 2.2], $\{E_s:s\in E_X\}$ is the collection of λ-classes that contain idempotents. Since $e,f\in E_s$ imply $\mathscr{I}(e)=\mathscr{I}(f)$, $E_s\subset E_S$ ($f\in\mathscr{I}(e)$ and $e_t\in E_S$ imply $f\in E_S$ by [6, lemma 1.3]). Since $(e,f)\in\mathscr{L}(\in E_S)$ implies $(e,f)\in\rho$, E_s is a union of $\mathscr{L}$-classes of E_S.

For each $s\in X$, select precisely one $s'\in\mathscr{I}(s)$. If $s\in E_X$, let $s'=s$.

LEMMA 4. $\lambda_p\subset\cup(L_f:f\in E_{p'p})$

PROOF. If $x\in\lambda_p$, $x\in R_e\cap L_f$ for some $e,f\in E_S$. Hence, by the proof of [1, theorem 2.18], there exists $x'\in R_f\cap L_e\cap\mathscr{I}(x)$ such that $xx'=e$ and $x'x=f$. Thus, $x'\in\lambda_{p*}$ for some $p^*\in\mathscr{I}(p)$ and $f\in E_{p^*p}=E_{p'p}$.

For each $s\in E_X$, select and fix an $\mathscr{L}$-class I_s of E_s and an $\mathscr{R}$-class J_s of E_s.

If $s \in X$, let u_s denote a representative element of λ_s. If $s \in E_X$, let $u_s = I_s \cap J_s$.

LEMMA 5. *Every element of S may be uniquely expressed in the form* $x = iu_s j$ *where* $i \in I_{ss'}$ *and* $j \in J_{s's}$.

PROOF. Let $x \in \lambda_s$. By lemma 3 and lemma 4, $(x, j) \in \mathscr{L}$ for some $j \in J_{s's}$. Since $u_{s'}u_s \in E_{s's}$, $x = xju_{s'}u_s j = (xu_{s'})(u_s j)$ by lemma 3. Since $xu_{s'} \in E_{ss'}$, $(xu_{s'}, i) \in \mathscr{R}$ for some $i \in I_{ss'}$. By the proof of lemma 4, $(u_s, w) \in \mathscr{R}$ for some $w \in E_{ss^*}$ where $s^* \in \mathscr{I}(s)$. Since $E_{s's}$ and E_{s^*s} are contained in the same $\mathscr{D}$-class of E_S, $x = i(xu_{s'})wu_s j = iu_s j$. Suppose that $x = iu_s j = pu_t q$ where $p \in I_{tt'}$ and $q \in J_{t't}$. Clearly, $s = t$. Since $(u_s, f) \in \mathscr{L}$ for some $f \in E_{s's}$ by lemma 4, $iu_s = pu_s$, and, hence, $iw = pw$ and $i = p$. Similarly, $j = q$.

LEMMA 6. $I_s I_t \subset I_t$ *if* $st = t$ *and* $J_s J_t \subset J_s$ *if* $st = s$.

PROOF. Let $e \in I_s$ and $f \in I_t$. Thus, $f = f(ef)f$ by lemma 3, and, hence, $(ef, f) \in \mathscr{L}(\in E_t)$ and $ef \in I_t$. For brevity, let $E = E_S$.

LEMMA 7. *There exist mappings* $(r, s) \to \alpha_{(r,s)}$ *and* $(r, s) \to \beta_{(r,s)}$ *of* X^2 *into* $\mathscr{T}_E$, *the full transformation semigroup on E, defined by* $u_r g_q u_s = g_q \alpha_{(r,s)} u_{rqs} g_q \beta_{(r,s)}$ *where* $g_q \in E_q$. *We have*

(a) $E_q \alpha_{(r,s)} \subset I_{(rqs)(rqs)'}$; $E_q \beta_{(r,s)} \subset J_{(rqs)'(rqs)}$

(b) $z\alpha_{(s,t)}((z\beta_{(s,t)}r)\alpha_{(st,g)}) = (z(r\alpha_{(t,g)}))\alpha_{(s,tg)}$ *and*
$(z\beta_{(s,t)}r)\beta_{(st,g)} = (z(r\alpha_{(t,g)}))\beta_{(s,tg)}(r\beta_{(t,g)})$ *for* $z \in E_{s'stt'}$ *and* $r \in E_{t'tgg'}$.

(c) $(iu_s j)(pu_t q) = i((jp)\alpha_{(s,t)})u_{st}(jp)\beta_{(s,t)}q$ *where* $i \in I_{ss'}$, $j \in J_{s's}$, $p \in I_{tt'}$, $q \in J_{t't}$, $i((jp)\alpha_{(s,t)}) \in I_{(st)(st)'}$, *and* $(jp)\beta_{(s,t)}q \in J_{(st)'st}$.

PROOF. The first part of the lemma and (a) follow directly from lemma 5. Applying the definitions of $\alpha_{(r,s)}$ and $\beta_{(r,s)}$ to "$(u_s z)(u_t r u_g) = (u_s z u_t)(r u_g)$" and, then utilizing lemma 6 and lemma 5, we obtain (b). To obtain (c), consider $i(u_s(jp)u_t)q$ and, then, apply lemma 6.

Let X be an $\mathscr{L}$-unipotent semigroup with semigroup of idempotents Y and let E be a band Y of rectangular bands $\{E_y : y \in Y\}$ (By [10, proposition 5], Y is a semilattice of right zero semigroups). For each $y \in Y$, select an $\mathscr{L}$-class I_y of E_y and an $\mathscr{R}$-class J_y of E_y. For each $s \in X$, select an inverse s' of s such that $s^2 = s$ implies $s = s'$. Let $(r, s) \to \alpha_{(r,s)}$ and $(r, s) \to \beta_{(r,s)}$ be mappings of X^2 into $\mathscr{T}_E$, the full transformation semigroup on E, subject to the conditions

I. $E_q\alpha_{(r,s)}\subset I_{(rqs)(rqs)'}$; $E_q\beta_{(r,s)}\subset J_{(rqs)'rqs}$

II. $z\alpha_{(s,t)}((z\beta_{(s,t)}r)\alpha_{(st,g)})=(z(r\alpha_{(t,g)}))\alpha_{(s,tg)}$ and

$(z\beta)_{(s,t)}r)\beta_{(st,g)}=(z(r\alpha_{(t,g)}))\beta_{(s,tg)}(r\beta_{(t,g)})$ for $z\in E_{s'stt'}$ and $r\in E_{t'tgg'}$.

Let (X,E,Y,I,J,α,β) denote $\{(i,s,j): s\in X, i\in I_{ss'}, j\in J_{s's}\}$ under the multiplication $(i,s,j)(p,t,q)=(i((jp)\alpha_{(s,t)}), st, (jp)\beta_{(s,t)}q)$.

THEOREM 8. *S is a regular semigroup whose idempotents form a semigroup if and only if* $S\cong(X,E,Y,I,J,\alpha,\beta)$ *for some collection* X,E,Y,I,J,α,β.

PROOF. Let S be a regular semigroup whose idempotents from a semigroup and let $\lambda=\rho^t$ denote the congruence given in lemma 2. Let $X=S/\lambda$ and $Y=E_X$. Thus, X is an $\mathscr{L}$-unipotent semigroup by lemma 2, and $E=E_S$ is a band of rectangular bands $\{E_y: y\in Y\}$ by lemma 3. The mappings $(r,s)\to\alpha_{(r,s)}$ and $(r,s)\to\beta_{(r,s)}$ of X^2 into $\mathscr{T}_E$, satisfying I and II are given by lemma 7 ((a) and (b)). By lemma 5 and lemma 7 (c), $(iu_sj)\varphi=(i,s,j)$ defines an isomorphism of S onto (X,E,Y,I,J,α,β). We next show that $T=(X,E,Y,I,J,\alpha,\beta)$ is a regular semigroup whose idempotents form a semigroup. We utilize I and the proof of lemma 6 to establish closure and II to establish associativity. Utilizing I, $E_T=\{(i,s,j): s\in Y,\ i\in I_s,\ j\in J_s\}$ and, hence, E_T is a semigroup since Y is a semigroup. If $(i,s,j)\in T$, $k\in I_{s'(s')'}$, and $n\in J_{(s')'s'}$, we obtain $(i,s,j)(k,s',n)(i,s,j)=(i,s,j)$ by utilizing I.

In closing, we give a second structure theorem for generalized $\mathscr{L}$-unipotent semigroups.

Let X be an $\mathscr{L}$-unipotent semigroup with semigroup of idempotents Y and let E be a band Y of left zero semigroups $\{E_y: y\in Y\}$. For each $s\in X$, select an inverse s' of s such that $s^2=s$ implies $s=s'$. Let $(r,s)\to\alpha_{(r,s)}$ be a mapping of X^2 into $\mathscr{T}_E$, the full transformation semigroup on E, and let $y\to f_y$ be a mapping of Y into E subject to the conditions

I′ $E_q\alpha_{(r,s)}\subset E_{(rqs)(rqs)'}$: $f_y\in E_y$

II′ $z\alpha_{(s,t)}((f_{(st)'st}r)\alpha_{(st,g)})=(z(r\alpha_{(t,g)}))\alpha_{(s,tg)}$ for $z\in E_{s'stt'}$ and $r\in E_{t'tgg'}$.

Let (X,E,Y,α,f) denote $\{(i,s): s\in X,\ i\in E_{ss'}\}$ under the multiplication $(i,s)(p,t)=(i((f_{s's}p)\alpha_{(s,t)}), st)$.

COROLLARY 9. (cf. Warne, [11]). *S is a generalized $\mathscr{L}$-unipotent semigroup if and only if* $S \cong (X, E, Y, \alpha, f)$ *for some collection* X, E, Y, α, F.

PROOF. Let S be a generalized $\mathscr{L}$-unipotent semigroup. By [11, lemma 1], S is regular and E_S is a semigroup. Utilizing [11, theorem 2 and lemma 3], each E_y in the statement of theorem 8 is a left zero semigroup.

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REFERENCES

[1] A.H. Clifford and G.B. Preston, *The algebraic theory of semigroups*, Vol. 1 (Math. Surveys of the Amer. Math. Soc. 7, Providence, R.I., 1961.)

[2] A.H. Clifford and G.B. Preston, *The algebraic theory of semigroups*, Vol. 2 (Math. Surveys of the Amer. Math. Soc. 7, Providence, R.I., 1967).

[3] T.E. Hall, *On regular semigroups whose idempotents form a semigroup*, Bull. Austral. Math. Soc. 1 (1969), 195—208.

[4] M. Gerard Lallement, *Congruences et equivalences de Green sur un demi-group regulier*, C.R. Acad. Sci Paris 262 (1966), 613—616.

[5] David McLean, *Idempotent Semigroups*, Amer. Math. Monthly 61 (1954), 110—113.

[6] N.R. Reilly and H.E. Schieblich, *Congruences on Regular Semigroups*, Pacific J. Math. 23 (1967), 348—360.

[7] R.J. Warne, *E-bisimple Semigroups*, Journal of Natural Sciences and Math. (W. Pakistan), 10, No. 2, October, 1970.

[8] R.J. Warne, *L-regular semigroups I*, Kyungpook Math. Journal, 10, No. 2 (1970), 97—103.

[9] R.J. Warne, *Simple L-regular semigroups*, The Formosan Science, 24, No. 3-4, (1970), 71—77.

[10] R.J. Warne, *$\mathscr{L}$-unipotent semigroups*, to appear.

[11] R.J. Warne, *Generalized $\mathscr{L}$-unipotent semigroups*, to appear.

[12] Miyuki Yamada, *On a regular semigroup in which the idempotents form a band*, Pacific J. Math., 33 (1970), 261—272.