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ON MODULES OVER (qa)-RINGS

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All rings considered will have a unit and modules will be unital left modules. In a recent paper [4], H. Harui has considered modules over commutative rings whose total quotient ring is Artinian, called (qa)-rings. The purpose of this note is to extend two of the main results of his paper to non-commutative rings. Specifically, we show (Theorem 2.2) that for a ring R having a quasi-Frobenius quotient ring, if M is an h-divisible R-module such that $M/t(M)$ is an injective R-module then $t(M)$ is a direct summand of M: this extends Theorem 2.10 of [4]. Moreover, extending Theorem 3.3 of [4], we show that if R is a ring with an Artinian quotient ring Q, then Q is semisimple if and only if $E(M)/M$ is a torsion R-module for all R-modules M.

1. Preliminaries.

Our notation will be the same as that of [4]. Thus let S denote the set of non-zero divisors of R. An R-module M is *divisible* if $aM=M$ for all $a\in S$, while M is *h-divisible* if M is a homomorphic image of an injective R-module. For any R-module $M, t(M)=\{x\in M \mid ax=0 \text{ for some } a\in S\}$. A ring R is quasi-Frobenius if R is (left) Artinian and R is an injective R-module. A ring Q is a quotient ring of R if $a^{-1}\in Q$ for all $a\in S$ and given any $q\in Q, q=a^{-1}b$ for some $a\in S, b\in R$. Finally, the (left) singular ideal of R will be denoted by $Z_l(R)$, while $E(M)$ will denote the injective envelope of the R-module M.

2. Main Results.

The following proposition is well-known and its proof is similar to that for commutative rings, (see, e.g. [6]).

PROPOSITION 2.1. *Let R be a ring with quotie.t ring Q. Then:*

(a) *Every injective R-module is divisible.*

(b) *Every torsion-free divisible R-module M is a Q-module and M is Q-injective if and only if M is R-injective.*

We now extend Theorem 2.10 of [4] to non-commutative rings having a quasi-

Frobenius ring of quotients.

THEOREM 2.2. *Let R be a ring having a quasi-Frobenius quotient ring Q. If M is an h-divisible R-module and $M/t(M)$ is an injective R-module, then $t(M)$ is a direct summand of M.*

PROOF. Since $M/t(M)$ is a torsion-free injective R-module, $M/t(M)$ is an injective Q-module by Proposition 2.1. As Q is quasi-Frobenius, every injective Q-module is a projective Q-module [1, p. 402]. Now M is h-divisible hence, as in [4], there is an R-module F which is isomorphic to a direct sum of copies of Q and an epimorphism $\alpha : F \to M$. Thus F is also a Q-module. Let $\beta : M \to M/t(M)$ be the natural homomorphism and let $f = \beta\alpha : F \to M/t(M)$. For any $q = a^{-1}b \in Q$, $x \in F$ we have $a[\alpha(qx) + t(M)] = \alpha(bx) + t(M) = b[\alpha(x) + t(M)]$ and so $f(qx) = qf(x)$; i.e., f is a Q-homomorphism. Thus $K = \text{Ker } f$ is a Q-direct summand of F, say $F = K \oplus D$, and this is also a splitting of F as an R-module. Now let $A = \alpha(D)$. We claim that $M = A \oplus t(M)$. For if $x \in A \cap t(M)$, $x = \alpha(d)$ then $0 = \beta\alpha(d) = f(d)$ so $d = 0$ and hence $x = 0$. Also if $m \in M$, $m = \alpha(x)$ with $x = d + k$ where $d \in D$, $k \in K$ and so $m = \alpha(d) + \alpha(k)$. Since $0 = f(k) = \beta\alpha(k)$, $\alpha(k) \in t(M)$ and so $M = A \oplus t(M)$. This completes the proof.

As a consequence we have the

COROLLARY. *Let R be a ring having a semisimple Artinian quotient ring Q. Then $t(M)$ is a direct summand of M for every h-divisible R-module M.*

PROOF. Since $M/t(M)$ is torsion-free and h-divisible (hence divisible) it is a Q-module and thus Q-injective. But then $M/t(M)$ is R-injective and the theorem applied.

We next consider a condition which ensures that a ring R having an Artinian quotient ring will be semiprime. This result extends Theorem 3.3 of [4] to non-commutative rings.

THEOREM 2.3. *Let R be a ring having an Artinian quotient ring Q. The following conditions are then equivalent:*

(a) R is semiprime.

(b) Q is semisimple.

(c) $E(M)/M$ is a torsion R-module for each R-module M.

PROOF. The equivalence of (a) and (b) follows from Small's Theorem [7] since $P(R) = R \cap P(Q)$ where $P(Q) - (P(R))$ denotes the prime radical of $Q(R)$.

If (b) holds then by [3, Thm. 3.9] every essential left ideal of R contains a non-zero divisor. Since M is essential in $E(M)$, for any $x\in E(M)$, $(M:x)=\{r\in R \mid rx\in M\}$ is an essential left ideal of R. Thus there is a non-zero divisor $a\in R$ for which $ax\in M$ and so $E(M)/M$ is a torsion R-module. Now assume (c) holds. First note that $t(E(R))=0$ since R is essential in $E(R)$ and $t(R)=0$. Thus $\mathrm{Hom}_R(E(R)/R, E(R))=0$ and so the exact sequence $0\to R\to E(R)\to E(R)/R\to 0$ gives rise to the exact sequence $0\to\mathrm{Hom}_R(E(R), E(R))\to\mathrm{Hom}_R(R, E(R))\to\mathrm{Ext}_R^1(R, E(R))=0$, and $\mathrm{Hom}_R(R, E(R))\approx E(R)$. Thus $E(R)$ has a ring structure compatible with the R-module structure. Moreover we can consider $Q\subset E(R)=K$ so R has K as a ring of quotients and it can be verified that K is self-injective. Since R is finite-dimensional as an R-module, K is finite-dimensional as a K-module. Now if I is an essential left ideal of R then $E(I)=E(R)$ so $R/I\subset K/I$ hence R/I is torsion. Thus every essential left ideal of R contains a non-zerodivisor and so $Z_l(R)=0$. But then $Z_l(K)=0$ and so by [2, Thm. 1, p. 44], K is a regular finite-dimensional ring hence semisimple Artinian. Since $P(R)=P(K)\cap R$, R is then semiprime. This completes the proof.

REMARK. We have shown above that $E(R)$ is the maximal left quotient ring of R so that an alternate proof could be given by appealing to results of R. E. Johnson [5].

COROLLARY. *Let R be a ring with a semisimple Artinian ring of quotients. If M is a torsion R-module then $E(M)$ is a torsion R-module.*

We conclude by giving a condition that ensures that a ring having an Artinian ring of quotients will have a quasi-Frobenius ring of quotients.

PROPOSITION 2.4. *Let R be a ring with an Artinian quotient ring Q. Then Q is quasi-Frobenius if and only if Q is an h-divisible R-module.*

PROOF. If Q is quasi-Frobenius then Q is an injective Q or R-module. Thus suppose Q is h-divisible and let A be an injective R-module mapping onto Q. Then for some index set I, $\oplus\sum_{i\in I}R_i(R_i=R$ for all $i\in I)$, maps onto A and this mapping can be extended to an R-homomorphism from $\oplus\sum_{i\in I}E(R_i)=F$ onto Q. As in theorem 2.2 $E(R)$ is a Q-module and this last mapping is a Q-homomorphism. Since Q is Q-projective $F=Q_0\oplus P$ as Q-modules, with $Q\approx Q_0$. Then Q_0, being cyclic, lies in $F_0=\oplus\sum_{j=1}^{k}E(R_{i_j})$ for some finite subset $\{i_1, \cdots, i_k\}\subset I$. Hence it follows that

$F_0 = Q_0 \oplus (P \cap F_0)$ and so Q_0 is R-injective and so Q is Q- and R-injective by Proposition 2.1.

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