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PAN-GEODESICS IN THE PSEUDO-RIEMANNIAN SPACE WITH SCHOUTENEAN CONNECTION

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[I] We had considered some properties in the semi-symmetric Schoutenean connection space whose metric and connection are given as follows:

$$ds^2=g_{ij}dx^i dx^j$$

and

$$\Gamma_i{}^h{}_j=\{{}_i{}^h{}_j\}+g_{ij}V^h-\delta_j^h V_i$$

where $\{{}_i{}^h{}_j\}$ is the Christoffel's symbol with respect to g_{ij} and V_i the torsion vector [1], [2].

Usually we would like to obtain the repère at the origin M:

$$\boldsymbol{dM}=dx^i\boldsymbol{e}_i$$

$$\boldsymbol{de}_i=\omega_i{}^k\boldsymbol{e}_k$$

where

$$\omega_i{}^k=\Gamma_i{}^k{}_h dx^h$$

Now, by the definition we will say that a curve is the pan-geodesics if the equation

$$\frac{\boldsymbol{d^2M}}{ds^2}=\lambda\frac{\boldsymbol{dM}}{ds}+\mu\boldsymbol{V}$$

holds, where $\boldsymbol{V}$ represents the torsion vector and ds the line element of the curve.

This relation, on the other hand, means that the osculating plane of the curve contains the torsion vector of the space. Multiplying by $\frac{\boldsymbol{dM}}{ds}$, we get

$$\lambda+\mu\boldsymbol{V}\frac{\boldsymbol{dM}}{ds}=0$$

and then this implies the following two cases: the first, if $\mu=0$ then $\lambda=0$, i.e., the pan-geodesics becomes astraight line and the second, if $\mu=1$ then,

$$\lambda=-\boldsymbol{V}\frac{\boldsymbol{dM}}{ds}$$

that is, the pan-geodesics should be reduced to the normal geodesics of the space.

We could have rewritten the equation of a pan-geodesics as follows;

$$(1)\qquad \frac{d^2x^i}{ds^2}+\left\{{}_{k}{}^{i}{}_{h}\right\}\frac{dx^k}{ds}\frac{dx^h}{ds}=(\mu-1)\left(V^i-\frac{dx^i}{ds}V_k\frac{dx^k}{ds}\right)$$

We put the following condition in the equation (1)

$$\mu-1=\psi'(U),$$

then this condition leads us to the problem on both the pan-geodesics and its trajectories. We study this in our pseudo-Riemannian space [2] of the metric

$$ds^2=e^{2U}g_{ij}dx^idx^j$$

and the torsion

$$V_i=\frac{\partial U}{\partial x^i}.$$

The well-known equation of movement of a unit mass point generated by the force F^i should be reformed as follows;

$$(2)\qquad \frac{d^2x^i}{dt}+\left\{{}_{k}{}^{i}{}_{h}\right\}\frac{dx^k}{dt}\frac{dx^h}{dt}+\frac{dx^i}{dt}\frac{\partial U}{\partial x^k}\frac{dx^k}{dt}=F^i$$

Here we consider the case of F^i satisfies

$$(3)\qquad F^i=a^2g^{ik}e^{2(\phi-U)}\psi'(U)\frac{\partial U}{\partial x^k}$$

Putting

$$(4)\qquad ds=vdt$$

then the equation (2) could be rewritten in the following form;

$$(5)\qquad v^2\left[\frac{d^2x^i}{ds^2}+\left\{{}_{k}{}^{i}{}_{h}\right\}\frac{dx^k}{ds}\frac{dx^h}{ds}+\frac{dx^i}{ds}\frac{\psi U}{\partial x^k}\frac{dx^k}{ds}\right]+\frac{dx^i}{ds}\cdot\frac{dv}{dt}$$

$$=a^2g^{ik}e^{2(\phi-U)}\psi'(U)\frac{\partial U}{\partial x^k}$$

and again, setting the following two conditions;

$$(6)\qquad \begin{cases} v^2-v_0{}^2=a^2\left(e^{2\phi(U)}-e^{2\phi(U_0)}\right) \\ v=ae^{\phi(U)} \end{cases}$$

Therefore, we have

$$v\frac{dv}{dt}=a^2e^{2\phi(U)}\psi'(U)\frac{\partial U}{\partial x^k}\frac{dx^k}{dt}$$

and in this case our equations (5) could take following form;

$$\frac{d^2x^2}{ds^2}+\left\{{}_{k}{}^{i}{}_{h}\right\}\frac{dx^k}{ds}\frac{dx^h}{ds}+\frac{dx^i}{ds}\frac{\partial U}{\partial x^k}\frac{dx^k}{ds}$$

$$=\psi'(U)\left[g^{ik}e^{-2U}\frac{\partial U}{\partial x^k}-\frac{\partial U}{\partial x^k}\frac{dx^k}{ds}\frac{dx^i}{ds}\right]$$

This also would be regard as the equation of a pan-geodesics. So that we might say that *in the pseudo-Riemannian space the trajectory of a point moving under the condition* (5) *is a pan-geodesics.*

[II] Next we consider the second differential form of the hypersurface;

$$\frac{d\sigma^2}{ds^2}=\nu_i\left[\frac{d^2x^i}{ds^2}+\left\{{}_k{}^i{}_h\right\}\frac{dx^k}{ds}\frac{dx^h}{ds}+V^i-V_k\frac{dx^k}{ds}\frac{dx^i}{ds}\right]$$

$$=\nu_i\left[\frac{d^2x^i}{ds^2}+\left\{{}_k{}^i{}_h\right\}\frac{dx^k}{ds}\frac{dx^h}{ds}+V^i\right]$$

$$=\kappa N_i\nu^i$$

where N_i be a unitary vector normal to the hypersurface and κ the curvature of a curve through the point (origin) M and ν^i the normalized vector of principal normal of this curve [1], [2].

In another form we set as follows;

$$\frac{d\sigma^2}{ds^2}=-\frac{dx^i}{ds}\left[\frac{dN_i}{ds}-\Gamma_i{}^h{}_kN_h\frac{dx^k}{ds}\right]$$

$$=-\frac{dx^2}{ds}\frac{dN_i}{ds}+\left\{{}_k{}^h{}_i\right\}N_h\frac{dx^i}{ds}\frac{dx^k}{ds}+V^hN_h$$

$$=\left(-\frac{\partial N_k}{\partial x^h}+\left\{{}_k{}^l{}_h\right\}N_l\right)\frac{dx^h}{ds}\frac{dx^k}{ds}+V^hN_h.$$

Hence, we have the second fundamental form of the hypersurface in the follows;

$$d\sigma^2=\left(-\frac{\partial N_k}{\partial x^h}+\left\{{}_k{}^l{}_h\right\}N_l\right)dx^hdx^k+V^hN_hds^2$$

According to the definition, the equation of the asymptotic curve (of the curve being contact of the second order) could be given by the following two conditions;

$$\left(-\frac{\partial N_k}{\partial x^h}+\left\{{}_k{}^l{}_h\right\}N_l\right)dx^kdx^h+V^kN_kds^2=0$$

and

$$N_i\nu^i=0.$$

We see this latter condition $N_i\nu^i=0$ requires that its osculating plane could be contained in the hyperplane tangent to the surface. Noting that the interior of the bracket in the left hand side of above the first condition be the covariant derivative of N_i with respect to $\left\{{}_j{}^i{}_k\right\}$, simply we set as f_{kh}, then finally we have the equation;

$$\kappa N_i \nu^i = f_{kh} \frac{dx^k}{ds} \frac{dx^h}{ds} + V^h N_h.$$

Hence we obtain the next proposition;

If the hyperplane tangent to a hypersurface contains the torsion vector of the space then this surface admits itself to be imbedded in the Riemannian space and particularly its asymptotic curve be the pan-geodesics of the space.

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