Bull. Korean Math. Soc.
Vol. 9 (1972), pp. 75-77

ON TOPOLOGICAL $\tilde{N}$-GROUPS, II

JAIHAN YOON*

1. We shall be concerned with the topological group G in which every closed maximal subgroup of any given closed subgroup is normal. We call such a group a *topological $\tilde{N}$-group*. By an N-group we mean a topological group that satisfies the normalizer condition for closed subgroups [**3**]. It is easy to see that every N-group is an $\tilde{N}$-group, and that every quotient group of an $\tilde{N}$-group is also an $\tilde{N}$-group in the topological sense. Moreover, every closed subgroup of an $\tilde{N}$-group is again an $\tilde{N}$-group. On the contrary, V. I. Ušakov has given an example of an N-group in which every closed subgroup is not necessarily an N-group [**3**]. Therefore, there exists an topological $\tilde{N}$-group which is not an N-group in the topological sense.

A simple example shows that there exists an $\tilde{N}$-group that is not an abstract $\tilde{N}$-group. In fact, let T be the usual cirle group, and let S be the symmetric group of four letters with topology defined by single normal subgroup A, the alternating subgroup of S. Define a map μ of S into the automorphism group $\mathrm{Aut}(T)$ of T by $\mu(\sigma)(t)=t^{\mathrm{sgn}\sigma}$ where $\mathrm{sgn}\sigma$ denotes 1 or -1 according as σ is even or odd permutation respectively. The map of $T\times S$ into T that sends each (t, σ) onto $\mu(\sigma)(t)$ is clearely continuous. Therefore the semidirect product $T\times_{\mu}S$ is a topological group with the product topology.

It is not hard to show that the group $T\times_{\mu}S$ is an $\tilde{N}$-group. By taking account of the existance of a maximal subgroup of S that is neither closed nor normal in S, we see that the group is not an abstract $\tilde{N}$-group.

2. In a topological group G, a normal system will mean a complete orderd set of closed subgroups of G. Similar arguments used in [**2**, p. 173] will prove the following lemma. Although, a slight modification due to the requirement of closedness of subgroups in a normal system is necessary.

LEMMA. *Every normal system of a topological group can be refined to a composition system.*

* Supported by a grant of the Ministry of Science and Technology.

With a help of the above Lemma, the following topological version of a theorem on a abstract $\tilde{N}$-group can be proved easily.

THEOREM 1. *A topological group is an $\tilde{N}$-group if and only if, for any closed subgroup, there is some normal system passing though it.*

Some of known properties of an N-group are based on the fact that the group, in particular, satisfies the normalizer condition for open subgroups. This is one of the reason for us to be interested in a suficient condition that makes an $\tilde{N}$-group to satisfy the condition for open subgroups.

THEOREM 2. *Let G be an $\tilde{N}$-group containing a dense subgroup D such that, for every closed subgroup in D, there passes through an ascending chain of closed subgroups of D admitting no further refinement. Then G satisfies the normalizer condition for open subgroups.*

Proof. Let H be an open proper subgroup of G. Since H is closed, the subgroup $H_1 = H \cap D$ is closed in D. Therefore, by the assumption on the group G, there is an ascending chain

$$H_1 \subset H_2 \subset \cdots \subset H_\tau = D$$

which admits no futher refinement. This implies

$$\bar{H}_1 \subset \bar{H}_2 \subset \cdots \subset \bar{H}_\tau = G$$

and $\bar{H}_1 = H$. Without loss of generality we may assume $\bar{H}_2 \neq H$. We shall show that H is a maximal subgroup of $\bar{H}_2$. Note that any group containing H is open. Suppose that $H \subset K \subset \bar{H}_2$ for some subgroup K of G. Then we have $H_1 = K \cap D$ or $H_2 = K \cap D$, and therefore $\overline{K \cap D} = H$ or $\bar{H}_2 = \overline{K \cap D}$. By taking account of that K is open, we have $K = H$ or $\bar{H}_2 = K$. Being G an $\tilde{N}$-group, the normalizer of H in G is different from H. This proves the theorem.

COROLLARY. *Let G be a group as in Theorem 2. If G contains an open compact subgroup, then the set of all compact elements of G is an open normal subgroup of G.*

Proof. The group G satisfies the normalizer condition for open subgroups. Therefore the same arguments in the proof of Theorem 2 of [3] applies.

THEOREM 3. *Let G be a topological group, and D a central subgroup of G. If G/D is an $\tilde{N}$-group, G is also an $\tilde{N}$-group provided that the natural map φ; $G \to G/D$ is closed.*

Proof. Let K be a closed subgroup of G and M a maximal closed subgroup of K; $K=\{x, M\}$. The assumption on the map φ implies that the subgroups $\varphi(K)$ and $\varphi(M)$ are closed in G/D. Clearly $\varphi(M)$ is maximal in $\varphi(K)$ or they coincide. Suppose that $\varphi(K)=\varphi(M)$. There exists an element m in M such that $xm^{-1}\in D$. Since the central element xm^{-1} can not belong to M, the subgroup generated by M and xm^{-1} coincides with K, and hence M is normal in K. For the remaining case, let k and m be in K and M respectively. There are some d in D and m_1 in M such that $kmk^{-1}=m_1d$, and therefore the element d must belong to M. If otherwise, $K=\{d, M\}$, which is impossible. Since the elements k and m were arbitrary, M is normal in K.

COROLLARY. *Let G be a compact simply connected topological group. If G_1 is a connected $\tilde{N}$-group and locally isomorphic to G, then G is an $\tilde{N}$-group.*

Proof. It is wellknown that G_1 is topologically isomorphic to a quotient group G/D by a discrete central subgroup D of G [1]. Clearly the map φ : $G\rightarrow G/D$ is continuous. Therefore, by Theorem 3, the group G is an $\tilde{N}$-group.

References

[1] C. Chevalley, *Theory of Lie groups*, Princeton University Press 1946.
[2] A.G. Kurosh, *The theory of groups*, Chelsea, 1955.
[3] V.I. Ušakov, *Groups with normalizer condition.* English transl., Amer. Math. Soc. Transl. (2) **82** (1969).

Seoul National University