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TOPOLOGICAL SPACES WITH CUSHIONED PAIR-SEMIDEVELOPEMENTS

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The main object of the present paper is to describe some properties of cushioned pair-semidevelopable spaces and certain relation between cushioned pair-semidevelopable spaces and stratifiable spaces.

J. G. Ceder introduced M_3-spaces [4] and C. J. R. Borges renamed them stratifiable spaces[3].

A topological space is a *stratifiable space* if, to each open set $U \subset X$, one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of open subsets of X such that

(a) $\mathrm{cl}\, U_n \subset U$,

(b) $\bigcup_{n=1}^{\infty} U_n = U$,

(c) $U_n \subset V_n$ whenever $U \subset V$.

A topoloical space is said to be *semi-developable* if there is a sequence of (not necessarily open) covers of X, $\gamma = \{\gamma_n\}_{n=1}^{\infty}$, such that for each $x \in X$, $\{\mathrm{st}(x, \gamma_n)\}_{n=1}^{\infty}$ is a neighborhood base at x. In this case, γ is called *semidevelopment* [1]. If γ and δ are collections of subsets of X, then we say that γ is *cushioned* in δ if one can assign to each $G \in \gamma$ a $D(G) \in \delta$ such that, for every $\gamma' \subset \gamma$, $\mathrm{cl}(\bigcup \{G | G \in \gamma'\}) \subset \bigcup \{D(G) | G \in \gamma'\}$. By a *cushioned pair-semidevelopment* for X we shall mean a pair of semi-developments (γ, δ) such that γ_n is cushioned in δ_n, for each n [2].

A space X is called *cushioned pair semi-developable* if and only if there exists a cushioned pair-semidevelopment for X.

The author quotes some result from [5] hereby as Lemma in order to use it for the proof of the following theorem.

LEMMA. *A space X is stratifiable if and only if to each closed subset $F \subset X$ one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of open subsets of X such that*

(a) $F \subset U_n$ *for each* n,

(b) $\bigcap_{n=1}^{\infty} (\mathrm{cl}\, U_n) = F$,

(c) $U_n \subset V_n$ *whenever* $U \subset V$.

THEOREM 1. *Every cushioned pair-semidevelopable space is stratifiable.*

Proof. Let X be a topological space with cushioned pair-semidevelopment (γ, δ) for X. For any closed subset $F \subset X$, let $U_n = \mathrm{Int}(\mathrm{cl}(\mathrm{st}(F, \gamma_n)))$. Then we have obviously (a) $F \subset U_n$ for each n. For the condition (b), assume that there exists a point $y \in \bigcap_{n=1}^{\infty} (\mathrm{cl} U_n) - F$. Since $y \in X - F$ and $X - F$ is open in X, there exists an integer m such that $\mathrm{st}(y, \delta_m) \subset X - F$. Therefore y does not belong to $\mathrm{st}(F, \delta_m)$.

$$\begin{aligned} \mathrm{cl} U_m &= \mathrm{cl}(\mathrm{Int}(\mathrm{cl}(\mathrm{st}(F, \gamma_m)))) \\ &\subset \mathrm{cl}(\mathrm{cl}(\mathrm{st}(F, \gamma_m))) = \mathrm{cl}(\mathrm{st}(F, \gamma_m)) \subset \mathrm{st}(F, \delta_m). \end{aligned}$$

But $\mathrm{st}(F, \delta_m)$ can not contain the point y, and hence $y \notin \mathrm{cl} U_m$. This contradicts to the assumtion, which implies $\bigcap_{n=1}^{\infty} (\mathrm{cl} U_n) - F = \phi$. Since it is clear that $\bigcap_{n=1}^{\infty} \mathrm{cl} U_n \supset F$, we have (b) $\bigcap_{n=1}^{\infty} \mathrm{cl} U_n = F$. It is easily shown that (c) $U_n \subset V_n$ if $U \subset V$. Hence X is a stratifiable space from the Lemma.

C. J. R. Borges show that every stratifiable space is normal[3]. This result and the above theorem give the following corollary.

COROLLARY. *Every cushioned pair-semidevelopable space is normal.*

THEOREM 2. *Any subset of a cushioned pair-semidevelopable space is cushioned pair-semidevelopable.*

Proof. Let X be a topological space with a cushioned pair semidevelopment (γ, δ) and $A \subset X$, For each i and for any subset $\gamma_i' \subset \gamma_i$,

$$\mathrm{cl}(\bigcup \gamma_i') \subset \bigcup \{D(G) \mid G \in \gamma_i'\}.$$

We define (γ^A, δ^A) by

$$\gamma^A = \{\gamma_i^A\}_{i=1}^{\infty}, \quad \gamma_i^A = \{A \cap G \mid G \in \gamma_i\}$$

and

$$\delta^A = \{\delta_i^A\}_{i=1}^{\infty}, \quad \delta_i^A = \{A \cap H \mid H \in \delta_i\}.$$

For any subset $\gamma'^A_i \subset \gamma_i^A$

$$\begin{aligned} \mathrm{cl}_A(\bigcup \gamma'^A_i) &= \mathrm{cl}_A(\bigcup \{A \cap G \mid G \in \gamma_i'\}) = \mathrm{cl}_A(A \cap \{\bigcup \{G \mid G \in \gamma_i'\}\}) \\ &\subset A \cap \mathrm{cl}(\bigcup \{G \mid G \in \gamma_i'\}) \\ &\subset A \cap (\bigcup \{D(G) \mid G \in \gamma_i'\}) = \bigcup \{A \cap D(G) \mid G \in \gamma_i'\}. \end{aligned}$$

Hence (γ^A, δ^A) is a cushioned pair-semidevalopment for A.

References

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