

## Development of algebraic geometry

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The main object of algebraic geometry is to study geometric properties of algebraic varieties. An algebraic variety  $V$  in an  $n$ -dimensional affine space is the set of solutions of polynomial equations  $F_1(x) = \cdots = F_r(x) = 0$ , where  $F_i(x) = F_i(x_1, \dots, x_n)$  are polynomials in the  $n$  variables  $x_1, \dots, x_n$ . In order to get a more interesting and more powerful geometry one introduces the points at infinity and considers algebraic varieties in a projective space. Then the equations take the form  $G_1(y) = \cdots = G_r(y) = 0$ , where  $G_i(y) = G_i(y_0, \dots, y_n)$  are homogeneous polynomials in  $n+1$  variables  $y_0, y_1, \dots, y_n$ .

Therefore one could say that algebraic geometry is just a higher-dimensional coordinate geometry. But if one sticks to the coordinates too closely, then one cannot go much further than the classical coordinate geometry on the plane or in the 3-space in which varieties defined by equations of low degrees were mainly considered, because in general the complexity of the equations increases very rapidly with the number of the variables and the degrees of the equations. So one needs more abstract viewpoint.

In the so-called abstract algebraic geometry the main tools are commutative algebra (theory of commutative rings and fields) and homological algebra (sheaf theory etc.). When one considers varieties defined over complex numbers, one can view them as complex analytic spaces and use analytic and topological methods (harmonic integrals, several complex variables etc.).

Conversely, algebraic geometry can be used in other branches of mathe-

matics. To take recent examples in analysis, Hironaka's resolution of real and complex-analytic spaces, and M. Artin's approximation theorem (of formal solution of analytic equations by analytic solution), would not have been possible without their knowledge of algebraic geometry, and Atiyah applied resolution of singularities to division of distributions (C.P.A.M. 23, 1970).

I do not know the old stories very well, but Riemann in the 19th century was perhaps the first man who considered birational geometry and obtained significant results. Two algebraic varieties  $V$  and  $V'$  are said to be birationally equivalent if there is a correspondence  $T$  between  $V$  and  $V'$  such that, if  $P \in V$  and  $T(P) = P' \in V'$  correspond to each other then the coordinates of  $P'$  are rational functions of the coordinates of  $P$  and vice versa. Thus  $T$  is one-to-one almost everywhere, except at the points where some denominators of the rational functions vanish. Let  $K(V)$  denote the field of rational functions of the coordinates on  $V$  (we assume that  $V$  and  $V'$  are irreducible). Then a birational correspondence  $T: V \rightarrow V'$  induces an isomorphism  $K(V) \cong K(V')$  over the constant field  $\mathbf{C}$ , and conversely such an isomorphism between the function fields determines a birational correspondence.

When  $V$  has complex dimension 1,  $K(V)$  is a field of algebraic functions of one variable, i.e. a finitely generated field of transcendence degree 1 over  $\mathbf{C}$ . Riemann proved that such a field is determined by a Riemann surface. More precisely, he found that

- (1) the isomorphism classes of smooth projective algebraic curves,
- (2) the isomorphism classes of compact complex manifolds of dimension 1, and
- (3) the isomorphism classes of fields of algebraic functions of one variable are essentially the same. In the higher-dimensional case this is no longer true. Two birationally equivalent smooth algebraic surfaces may

not be (biregularly) isomorphic, and there are compact complex analytic surfaces which are not algebraic. Still the function field  $K(V)$  is very important.

After Riemann, German mathematicians in the 19th century continued to develop the theory of algebraic curves. Max Noether (father of Emmy Noether) investigated algebraic curves in a 3-dimensional projective space and made a huge list of classification of such curves. At the same time he made some pioneering work on algebraic surfaces (*Zur Grundlegung der Theorie der algebraischen Raumkurven*, 1883). In France Poincaré and Picard developed an analytic theory of algebraic surfaces. But a decisive progress in algebraic geometry was done by Italian geometers such as Bertini, Castelnuovo, Enriques and Severi from the end of last century to the beginning of this century. They used divisors systematically to study birational geometry of algebraic surfaces.

In general, a divisor  $D$  on a smooth projective variety  $V$  of dimension  $n$  is a linear combination

$$D = \sum_{i=1}^g n_i W_i \quad (n_i \in \mathbb{Z})$$

of irreducible subvarieties  $W_i$  of codimension 1 (i.e. of dimension  $n-1$ ). A divisor is said to be positive if all the coefficients are positive. Each rational function  $\varphi \in K(V)$ ,  $\varphi \neq 0$ , determines a divisor  $(\varphi) = \sum n_i W_i - \sum m_j W'_j$ , where  $W_i$  is a zero of order  $n_i$  and  $W'_j$  is a pole of order  $m_j$  of  $\varphi$ . Two divisors  $D$  and  $D'$  are said to be linearly equivalent (notation:  $D \sim D'$ ), if there exists  $\varphi \in K(V)$  such that  $D - D' = (\varphi)$ . A complete linear system  $L = |D|$  is the set of the positive divisors which are linearly equivalent to  $D$ :

$$L = \{D' \mid D' \sim D, D' \geq 0\}.$$

It has a natural structure of a projective space. A subset  $M$  of a complete linear system  $L$  is called a linear system if  $M$  is a linear subspace of the projective space  $L$ . Linear systems are associated with rational maps of  $V$

into a projective space in the following way. The set of the hyperplanes  $\{H\}$  of a  $d$ -dimensional projective space  $\mathbf{P}_d$  is a complete linear system, and if  $F: V \rightarrow \mathbf{P}_d$  is a rational map of  $V$  into  $\mathbf{P}_d$  then the pull-back  $\{F^{-1}(H) | H \in \mathcal{L}\}$  of the system  $\{H\}$  is a linear system on  $V$  without fixed components. Conversely, if  $M$  is any linear system on  $V$ , if  $\dim M = r$  and if  $D_0, D_1, \dots, D_r$  are linearly independent elements of  $M$  with respect to its structure as a projective space, then choose  $\varphi_i \in K(V)$  ( $i=0, 1, \dots, r; \varphi_0=1$ ) such that  $D_i - D_0 = (\varphi_i)$  and consider the rational map  $F: V \rightarrow \mathbf{P}_r$  defined by  $F(x) = (\varphi_0(x) : \varphi_1(x) : \dots : \varphi_r(x))$ . Then  $F$  is determined by  $M$  up to projective transformations in  $\mathbf{P}_r$ , and the linear system  $\{F^{-1}(H)\}$  is precisely the linear system  $M'$  obtained from  $M$  by removing the fixed components from each member of  $M$ . If  $M'$  has no base points (=points belonging to all divisors in the linear system), then the rational map  $F$  is everywhere regular. If  $P$  is a base point of  $M'$  then it is "blown up" by  $F$ , that is, it corresponds to a subvariety of dimension  $\geq 1$  of  $F(V)$ .

Italian geometers used linear systems very effectively and developed a quite original method. Combining it with analytic methods they obtained classification of algebraic surfaces. But their tools were sometimes not sharp enough to give a rigorous proof. Gradually their papers became unreadable to the students, and the tradition of the Italian school came to an end in Italy. The necessary foundations of their method were built later by v. d. Waerden, Chevalley, Weil and Zariski by using modern algebra.

Now we come to the 20th century. Modern abstract algebra was founded by Emmy Noether, Emil Artin and others in the twenties of this century. In the theory of commutative rings E. Noether formulated the famous ascending chain condition and laid the foundation of ideal theory of noetherian rings. She also defined the notion of specialization in algebraic geometry, which was developed later by v.d. Waerden and André Weil.

The theory of noetherian rings was made big and powerful by the hands of W. Krull in 1920—1940. Among other things he built the dimension theory of ideals, showing the close relationship between algebraic geometry and noetherian ring theory. We will explain it in more detail.

Let  $k$  be an algebraically closed field, and let  $\mathbf{A}_n$  be an affine space over  $k$  of dimension  $n$ . Then there is a one-to-one correspondence between the set of the prime ideals of the polynomial ring  $k[X_1, \dots, X_n]$  and the set of the irreducible varieties in  $\mathbf{A}_n$ . (This is known as the zero-point theorem of Hilbert.) Let  $I$  be an ideal of  $k[X_1, \dots, X_n]$  and let  $\mathfrak{p}_1, \dots, \mathfrak{p}_r$  be the minimal prime ideals containing  $I$ ; let  $V(I)$  denote the variety of the zeros of  $I$ , namely  $V(I) = \{x \in \mathbf{A}_n \mid f(x) = 0 \text{ for all } f \in I\}$ . Then  $V(I) = V(\mathfrak{p}_1) \cup \dots \cup V(\mathfrak{p}_r)$ , and the right hand side gives the decomposition of  $V(I)$  into the irreducible components.

In general, let  $R$  be a commutative ring and  $\mathfrak{p}$  a prime ideal. By the height of  $\mathfrak{p}$ ,  $ht(\mathfrak{p})$ , we understand the maximum of the lengths of prime chains in  $\mathfrak{p}$ : if  $\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \dots \supset \mathfrak{p}_m$  is a longest chain of prime ideals of  $R$  contained in  $\mathfrak{p}$  then we put  $ht(\mathfrak{p}) = m$ . In the case of  $R = k[X_1, \dots, X_n]$ ,  $ht(\mathfrak{p})$  is equal to the codimension of the variety  $V(\mathfrak{p})$ , i.e.  $ht(\mathfrak{p}) = n - \dim V(\mathfrak{p})$ .

Krull proved the following fundamental

**THEOREM.** *If  $R$  is a noetherian ring, if  $I = a_1R + \dots + a_mR$  is an ideal generated by  $m$  elements and if  $\mathfrak{p}$  is a minimal prime ideal containing  $I$ , then  $ht(\mathfrak{p}) \geq m$ .*

Since  $\mathfrak{p}$  itself is generated by a finite number of elements, the theorem implies in particular that  $ht(\mathfrak{p})$  is always finite, so that the descending chain condition holds for the set of prime ideals in a noetherian ring. Geometrically, the theorem translates the fact that, if you add one more equation, then the dimension of the variety decreases by one at most.

From the ring  $R$  and the prime ideal  $\mathfrak{p}$  one constructs the local ring  $R_{\mathfrak{p}}$ . A noetherian ring is called a local ring if it has only one maximal ideal.

In the case  $R$  is an integral domain,  $R_{\mathfrak{p}}$  is just the subring of the field of fractions of  $R$  consisting of the elements  $a/b$  such that  $b \notin \mathfrak{p}$ . If  $R = k[X_1, \dots, X_n]/I$  and  $\mathfrak{m}$  is a maximal ideal corresponding to a point  $P$  of  $V = V(I)$ , then  $R_{\mathfrak{m}}$  is called the local ring of the point  $P$  on the variety  $V$ , and is denoted by  $\mathfrak{o}_P$  or  $\mathfrak{o}_{V,P}$ . It is simply the ring of the rational functions on  $V$  which are regular at  $P$ . The local ring  $\mathfrak{o}_P$  determines the variety  $V$  in the neighborhood of  $P$ , in the sense that if  $\mathfrak{o}_{V,P} \cong \mathfrak{o}_{V',P'}$ , then a suitable neighborhood  $U$  of  $P$  in  $V$  and a suitable neighborhood  $U'$  of  $P'$  in  $V'$  are isomorphic. The same is true for analytic varieties if we take the local ring of the analytic functions around the point.

Let  $R$  be a local ring,  $\mathfrak{m}$  its maximal ideal and  $k = R/\mathfrak{m}$ . The module  $\bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}$  has a natural structure of a graded ring containing the field  $k$ , and as such it is denoted by  $\text{gr}(R)$ . If  $\mathfrak{m}$  is generated by  $s$  elements then  $\text{gr}(R)$  is a homomorphic image of  $k[Y_1, \dots, Y_s]$ .

The function  $\sigma(\nu) = \sum_{i=0}^{\nu} \dim(\mathfrak{m}^i/\mathfrak{m}^{i+1})$ , where  $\dim$  denotes the dimension of vector space over  $k$ , is a polynomial in  $\nu$  for large values of  $\nu$ :  $\sigma(\nu) = a_0\nu^d + a_1\nu^{d-1} + \dots + a_d$  ( $\nu > \nu_0$ ). This polynomial is called the Hilbert polynomial of  $R$ . The degree  $d$  is equal to the dimension of  $R$ , i.e.  $ht(\mathfrak{m})$ . The number  $e = d!a_0$  is a positive integer and is called the multiplicity of the local ring  $R$ . The theory of multiplicity based on this definition is due to a later work of P. Samuel. When  $\text{gr}(R)$  is a polynomial ring over  $k$ ,  $R$  is called a regular local ring; geometrically it corresponds to a simple point of a variety. Krull showed that regular local ring has many good properties similar to the formal power series ring  $k[[X_1, \dots, X_d]]$ .

Krull also created the theory of general valuations, which has been applied to algebraic geometry by Zariski and Nagata. Krull confined himself to commutative algebra and did not discuss algebraic geometry. But his works inspired many algebraic geometers, and his theorems are at the basis of algebraic geometry of today.

It is Oscar Zariski who introduced modern algebra to algebraic geometry in late 1930s. In Japan, Yasuo Akizuki started as an algebraist and studied and generalized Krull's results, and turned to algebraic geometry partly under the influence of Zariski and Weil. Both Zariski and Akizuki promoted algebraic geometry in their countries, and they have had good students. A. Seidenberg, I. S. Cohen, S. Abhyankar, D. Mumford and M. Artin were Zariski's students; T. Matsusaka, Y. Nakai and S. Nakano were Akizuki's students, and J.-I. Igusa and M. Nagata were his young colleagues. Hironaka was first a student of Akizuki, and then of Zariski.

Before talking about Zariski we will briefly discuss the works of S. Lefschetz and B.L. van der Waerden. In 1920s Lefschetz studied the topology of, and the integrals on, algebraic varieties, in particular algebraic surfaces. His proofs were often too intuitive, but even today his works continue to inspire mathematicians. His results have been given rigorous proofs, and have been generalized, by Kodaira-Spencer, Akizuki-Nakano, A. H. Wallace, Bott, Grothendieck and his collaborators, and others. Lefschetz later turned to algebraic topology and, among other things, found his famous fixed point formula, which inspired A. Weil in his works and conjectures about congruence zeta functions.

Van der Waerden is one of the earliest who tried to provide algebraic geometry with solid foundations. He proved the triangulability of real or complex algebraic varieties and gave a topological foundation to the so-called enumerative geometry. He proved the general Bézout theorem by defining the intersection multiplicity geometrically. Moreover, the notion of the "associated form" of an algebraic variety, invented by W. L. Chow and v. d. Waerden, is of prime importance in the geometry in projective spaces. But v. d. Waerden used projective method rather than the advanced ideal theory of Krull, and so his works were not revolutionary and his book "Einführung in die Algebraische Geometrie" (1939) failed to

arouse interest widely.

Oscar Zariski was born in Russia, studied in Italy and worked in United States. At first he was a brilliant member of the Italian school and wrote the famous survey book "Algebraic Surfaces" (Ergebnisse, 1935). Around 1936 he intensively studied Krull's works and suddenly changed from a geometer to an ardent advocate of algebraic method in algebraic geometry. He was not satisfied with giving algebraic proofs to known results; rather, he used algebra to discover entirely new properties and to solve difficult old problems.

He showed the importance of normal varieties and defined the process of normalization. A point  $P$  on a variety  $V$  is called a normal point if the local ring  $\mathcal{O}_P$  is normal (i.e. is an integrally closed integral domain). A variety  $V$  is said to be normal if every point is normal. Zariski showed that, for any irreducible variety  $V$ , there exist a normal variety  $V^*$  and a regular birational map  $V^* \rightarrow V$  such that each point of  $V$  corresponds to a finite number of points of  $V^*$ . Such  $V^*$  is unique up to isomorphisms and is called the normalization (or the derived normal model) of  $V$ . A normal variety has no singularities of codimension one. Moreover, Zariski found that normal points have a very nice property with respect to birational transformations (the so-called Zariski Main Theorem).

Secondly, he applied successfully Krull's general valuation theory to the theory of birational transformations, and in particular to the resolution of singularities. In the weak form, resolving the singularities of a variety  $V$  implies finding a projective variety which is birationally equivalent to  $V$  and has no singular points. In the strongest form (proved by Hironaka in 1962) the "non-singular model" is to be obtained from  $V$  by a succession of transformations of a particularly good type. Zariski solved the resolution problem for dimension two, and then for dimension three (1944).

In 1943-45 C. Chevalley studied local rings, in particular their comple-



tions, and defined intersection multiplicities of algebraic and algebroid varieties. Let  $P$  be a point on an algebraic variety  $V$  and let  $\mathfrak{o}=\mathfrak{o}_P$  be its local ring. When the ground field  $k$  is the complex number field,  $V$  is also a complex number field,  $V$  is also a complex analytic space and  $\mathfrak{o}$  is a subring of the ring  $\mathfrak{o}^h$  of the holomorphic functions around  $P$ ; if  $V$  is embedded in  $A_n$  and is defined by an ideal  $I$  of  $k[X_1, \dots, X_n]$ , and if  $P$  is the origin  $(0, \dots, 0)$ , then  $\mathfrak{o}^h=k\{X\}/Ik\{X\}$ , where  $k\{X\}$  denotes the ring of the convergent power series in  $X_1, \dots, X_n$ . Let  $\mathfrak{m}$  be the maximal ideal of  $\mathfrak{o}$ . Then the local ring  $\mathfrak{o}$  is a metric space with respect to the  $\mathfrak{m}$ -adic topology (the powers  $\mathfrak{m}^i$  of  $\mathfrak{m}$  are taken as a fundamental system of neighborhoods of zero), and its completion  $\mathfrak{a}$  is a ring containing  $\mathfrak{o}$  and is isomorphic to  $k[[X]]/Ik[[X]]$ . we have  $\mathfrak{o} \subset \mathfrak{o}^h \subset \mathfrak{a}$ , and  $\mathfrak{a}$  is also the completion of  $\mathfrak{o}^h$  with respect to the  $\mathfrak{m}\mathfrak{o}^h$ -adic topology. When  $k$  is an abstract field  $\mathfrak{o}^h$  has no meaning, but the completion  $\mathfrak{a}$  still has a meaning. In the analytic case the completion  $\mathfrak{a}$  is very close to  $\mathfrak{o}^h$ , and so it is used to transport some analytic notions to the abstract case. Moreover,  $\mathfrak{a}$  is ring-theoretically simpler than  $\mathfrak{o}$ ; for example, if  $P$  and  $P'$  are simple points on varieties  $V$  and  $V'$  of the same dimension  $d$ , then  $\mathfrak{a}_P$  and  $\mathfrak{a}_{P'}$  are isomorphic (both  $\cong k[[X_1, \dots, X_d]]$ ) while  $\mathfrak{o}_P$  and  $\mathfrak{o}_{P'}$  are usually not isomorphic. A prime ideal in  $k[[X_1, \dots, X_n]]$  is said to define an algebroid variety. Chevalley's theory of multiplicity was simplified by P. Samuel in his thesis (1951) using Hilbert polynomials. Also Zariski made important contributions to the theory and application of completions and we shall come back to this later.

Andre Weil's epoch-making book "Foundations of Algebraic Geometry" was published in 1946. This book was algebraic, self-contained and rigorous. Algebraic geometry over an arbitrary ground field (in particular over a field of characteristic  $p$ ) had never been discussed with such thoroughness. Weil developed an intersection theory with special attention to the case of

characteristic  $p$ . Also, the notion of abstract variety (obtained by piecing affine varieties together) was introduced in this book for the first time ; so far only projective and affine varieties had been considered. The existence of complete non-projective varieties was proved later by Nagata.

Weil is above all a number theorist. He saw that Severi's theory of algebraic correspondence between algebraic curves would lead to a proof of the Riemann hypothesis for curves over finite fields. In order to establish Severi's theory in characteristic  $p$  he had to write Foundations. The function fields of algebraic curves over finite fields have similar properties with algebraic number fields, and their arithmetic properties had been investigated by E. Artin, F. K. Schmidt, H. Hasse, M. Deuring and others. In particular Hasse proved the Riemann hypothesis (about congruence zeta functions) for curves of genus 1 in 1936. The complete proof of the general case, by Weil, appeared in his second book "Sur les courbes algébriques..." (1948) which treated the theory of correspondences. In the third of the triplet, "Variétés abéliennes et courbes algébriques" (1948), he built the abstract theory of abelian varieties, which has become an important tool of abstract algebraic geometry. Weil declared himself to stand in the tradition of Kronecker (as opposed to that of Dedekind); not only he aimed at the unification of algebraic geometry and number theory, but also he employed very skillfully the Kroneckerian technique of using indeterminates. For ten years or more after 1946, many algebraic geometers wrote papers in Weil's language, quoting his theorems. But few could master his technique, and new methods of ideal theory and of homological algebra have gradually taken over the methods of Weil.

In the analytic and the topological theories, Hodge's harmonic integral (1941) and Leray's sheaf cohomology have become powerful tools in algebraic geometry. K. Kodaira, D. C. Spencer, J.-P. Serre and F. Hirzebruch successfully applied these methods to the proof of classical theorems of

Lefschetz and others, to various generalizations of the Riemann-Roch theorem, to finding new invariants of complex manifolds and to the classification of (algebraic as well as nonalgebraic) compact complex surfaces.

J.-P. Serre introduced sheaf theory and homological algebra in abstract algebraic geometry. His *FAC* (= *Faisceaux Algébriques Cohérents*, Ann. of Math. 61, 1955) marked a new epoch. In this work an algebraic variety over an algebraically closed field is viewed as a ringed space, i. e. a topological space (with respect to the Zariski topology in which the only closed sets are the subvarieties) with a sheaf of local rings. Classical invariants such as arithmetic genus are shown to be of cohomological nature. In another famous paper *GAGA* (= *géométrie algébrique and géométrie analytique*) he proved that, for projective varieties over the complex number field, the theory of analytic coherent sheaves and the theory of algebraic coherent sheaves are essentially the same. He also gave a cohomological characterization of regular local rings, expressed intersection multiplicity as the Euler-Poincaré characteristic of the Tor groups, and defined the important notion of flatness.

Serre's work was immediately generalized by A. Grothendieck to the grandiose theory of schemes (cf. his talk in Proc. Intern. Congr. Math. 1958). Take an equation  $f(X_1, \dots, X_n) = 0$  with coefficients in a commutative ring  $k$ . One can fix a commutative  $k$ -algebra  $B$  which is large enough for one's purpose and look for the solutions of  $f(X) = 0$  with coordinates in  $B$ . Alternatively, one may consider solutions in various  $B$ . In the latter viewpoint the equation defines, so to speak, a frame in which one may put various pictures. More precisely, put  $A = k[X_1, \dots, X_n]/(f)$ . Then the solutions with coordinates in  $B$  correspond to the  $k$ -algebra homomorphisms  $A \rightarrow B$ . Thus we get a covariant functor  $B \rightarrow X(B) = \text{Hom}_{k\text{-algebra}}(A, B)$ . This functor is an affine scheme (over  $k$ ). In general, any commutative ring  $A$  defines an affine scheme  $B \rightarrow \text{Hom}(A, B)$ . To be more geometric, one defines

an affine scheme  $\text{Spec}(A)$  as the ringed space of which the underlying topological space is the set of the prime ideals in  $A$  with Zariski topology (the sets  $D(a) = \{\mathfrak{p} \in \text{Spec}(A) \mid a \notin \mathfrak{p}\}$ ,  $a \in A$ , forming a base of the topology) and the structure sheaf  $\mathcal{O}$  is such that the stalk over a point  $\mathfrak{p}$  is the local ring  $A_{\mathfrak{p}}$ . The morphisms from  $\text{Spec}(B)$  to  $\text{Spec}(A)$  is defined to be the morphisms as local-ringed spaces. Then it turns out that there is a functorial isomorphism

$$\text{Hom}(\text{Spec}(B), \text{Spec}(A)) \cong \text{Hom}(A, B).$$

A scheme is defined to be a ringed space each point of which has an open neighborhood isomorphic to an affine scheme.

Thus the notion of a scheme is quite general. One important feature is that the structure sheaf may contain nilpotent elements; this is useful when one considers infinitesimal structure, and sometimes it enables one to use the method of successive approximation in abstract algebraic geometry; moreover, the fibre product (which is given by tensor product of rings in the case of affine schemes) can be used as a convenient substitute for the intersection product of Weil in many cases, which cannot happen if nilpotent elements are banned from the structure sheaf.

The generality of the notion of scheme also allows one to consider geometry over a ring instead of a field. Unification of number theory and algebraic geometry, once dreamt of by Kronecker, is partially realized by scheme theory so long as archimedean valuations can be neglected.

Grothendieck has brought in algebraic geometry many other revolutionary ideas, the impact of which are felt in other branches of mathematics also. The most salient feature is his functor-theoretic approach. For instance he gave a completely new formulation to the Riemann-Roch-Hirzebruch theorem (edited by Borel-Serre, 1958), introducing a new functor  $K(X)$ , which was subsequently developed by topologists into the so-called  $K$ -theory or extraordinary cohomology. He also gave a new construction of the Picard

variety by considering the Picard functor, which, in the words of S. Lang, marked the first complete separation from the tradition of the Italian school.

The problem of moduli for curves of given genus  $g$  is to find a good parametrization of the set of the isomorphism classes of non-singular algebraic curves of genus  $g$ . Already Riemann considered the problem and found that the correct number of parameters is 0 for  $g=0$ , 1 for  $g=1$  and  $3g-3$  for  $g>1$ . After an important but incomplete work of Severi, D. Mumford succeeded in constructing a good moduli scheme for such curves (*Geometric Invariant Theory*, Ergebnisse 1965). On the other hand, the similar problem for higher-dimensional compact complex manifolds was treated for the first time by a great work of K. Kodaira -D. C. Spencer as *the problem of deformation of complex structures* (Ann. of Math. 1958), the influence of which can be seen in Grothendieck's *EGA* also. If  $V$  is a compact complex manifold and  $\theta$  is the tangential sheaf of  $V$  (=the sheaf of germs of holomorphic tangent vectors), then the first approximation of the set of small deformations of  $V$  is given by  $H^1(V, \theta)$ ; M. Kuranishi proved the existence of the local deformation space for  $V$  (Ann. of Math. 1962), the dimension of which is in general  $\leq \dim H^1(V, \theta)$ . The similar local problem in the case of algebraic schemes with singularities has been successfully investigated by D. S. Rim (to appear in Publ. I.H.E.S.). Let me remark that  $H^0(V, \theta)$  (=the vector space of the global tangent vector fields on  $V$ ) is the tangent space at the automorphism group scheme  $Aut(V)$ , which was constructed by F. Oort and myself. Automorphism is much easier than deformation.

In the recent development I will mention the names of three distinguished disciples of Zariski, namely D. Mumford, H. Hironaka and M. Artin (son of Emil Artin), and a differential geometer P. Griffiths. Mumford has obtained deep results on abelian varieties and constructed an abstract

theory of theta functions. Hironaka's great work on resolution of singularities of algebraic and analytic varieties has been used by Grothendieck, Griffiths and others as a basis of new theories. Griffiths is doing a remarkable work about the algebraic cycles on an algebraic manifold over  $\mathbb{C}$ . So far we had a satisfactory theory only for divisors, i.e. cycles of codimension 1.

I will devote the rest of this exposition to one particular side of abstract algebraic geometry, i.e. the effort to describe analytic properties in terms of purely algebraic concepts. One approach is, as already said, by completion. If a variety  $V$  over  $\mathbb{C}$  decomposes into several analytic sheets in the neighborhood of a point  $P$ , then these sheets correspond to the minimal prime ideals of  $\mathfrak{o}_P^h$ ; that they also correspond to the minimal prime ideals of  $\mathfrak{o}_P$  had been conjectured, and a rigorous proof was given by Nagata in 1953 as an application of his theory of Henselian rings. He showed that a prime ideal in the convergent power series ring remains prime in the formal power series ring.

Zariski proved that if a local ring  $\mathfrak{o}$  of a variety is normal then  $\mathfrak{o}$  (hence also  $\mathfrak{o}^h$  if the ground field is  $\mathbb{C}$ ) is again normal (1950). This "analytic normality" is not true for general local rings, so it is natural to look for a good class of noetherian rings for which the theorem of analytic normality holds. This problem has been answered by Grothendieck's theory of excellent rings (EGA Ch.IV). Zariski also invented the theory of abstract holomorphic functions on a variety  $V$  along a subvariety  $W$ , and applied it to the algebraic proof of the principle of degeneration, which says that any specialization of a connected positive cycle in a projective space is connected. The theory was developed by Grothendieck as the theory of formal schemes. If the subvariety  $W$  is defined by a coherent sheaf of ideals  $I$  of the structure sheaf  $\mathcal{O}=\mathcal{O}_V$  of  $V$ , then the projective limit  $\hat{\mathcal{O}}=\varprojlim \mathcal{O}/I^n$  is a sheaf of local rings with support  $W$ . The local ringed space

$\hat{V}=(W, \hat{\sigma})$  is called the formal completion of  $V$  along  $W$ , and the ring of holomorphic functions of Zariski is nothing but  $H^0(W, \hat{\sigma})$ . Intuitively,  $V$  is something like the limit of shrinking analytic neighborhoods of  $W$  in  $V$ . Grothendieck built the cohomology theory of formal schemes and proved Zariski's theorems about  $H^0(W, \hat{\sigma})$  by descending induction on  $H^1(W, \hat{\sigma})$ . Hironaka studied formal meromorphic functions (which are obtained from  $\hat{\sigma}$  in the usual way) and proved that, if  $V$  is a projective space and  $W$  is a connected subvariety of positive dimension, then the field  $K(\hat{V})$  of the formal meromorphic functions is equal to the function field  $K(V)$ ; this implies, in particular, that any meromorphic function in a connected neighborhood (in the ordinary topology) of  $W$  in a complex projective space is extendable to a meromorphic function in the whole projective space. The result has been generalized to wider classes of pairs of a variety and a subvariety by Hartshorne, Hironaka and myself. Here, analytic theorems are obtained from stronger theorems in formal geometry, in which one need not worry about convergence.

Another approach is using étale neighborhoods. A morphism  $f: U \rightarrow V$  of schemes is said to be *étale* if it is flat and unramified; when non-singular algebraic varieties over  $\mathbf{C}$  are concerned it is equivalent to say that  $f$  induces a local isomorphism of the associated analytic manifolds. If a point  $P$  of  $V$  is contained in the image  $f(U)$  then  $f: U \rightarrow V$  is called an étale neighborhood of  $P$  in  $V$ . A family of étale morphisms  $f_i: U_i \rightarrow V$  such that  $\cup_i f_i(U_i) = V$  is called an étale covering. Zariski topology of  $V$  is sometimes too coarse to pursue analogies of analytic theories, and the use of étale coverings was initiated by Serre in the theory of algebraic fibre bundles. If  $E \rightarrow V$  is not locally trivial with respect to the Zariski topology on  $V$ , it may happen that there exists an étale covering  $f_i: U_i \rightarrow V$  such that  $f_i^{-1}(E) = E \times_V U_i$  are isomorphic to  $F \times U_i$ , and then  $E$  is called an isotrivial fibre bundle with fibre  $F$  (or an algebraic fibre bundle with respect to the

étale topology). It is clear that such  $E$  is an analytic fibre bundle over  $V$ . Grothendieck used étale coverings to define a new cohomology theory. This idea was developed by S. Lubkin, Grothendieck and M. Artin and applied to congruence zeta functions of higher-dimensional varieties.

These two approaches have been united in the recent work of M. Artin. His approximation theorem in the analytic case (Invent. Math. 1968) asserts the following. *Let  $K$  be a valued field of characteristic zero (e.g.  $\mathbf{R}$ ,  $\mathbf{C}$  or the  $p$ -adic number field) and consider a system of analytic equations*

$$(*) f(x, y) = 0, \quad f = (f_1, \dots, f_m)$$

where  $f_i(x, y)$  are convergent power series in the variables  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_N)$  with coefficients in  $k$ . Suppose that

$$\bar{y}(x) = (\bar{y}_1(x), \dots, \bar{y}_N(x)), \quad \bar{y}_j(x) \in k[[x]]$$

are formal power series without constant term which solve the equations (\*), i.e. such that  $f(x, \bar{y}(x)) = 0$ . Let  $c$  be an integer. Then one can find a convergent power series solution  $y(x) = (y_1(x), \dots, y_N(x))$  such that  $y_j(x) - \bar{y}_j(x)$  ( $j = 1, \dots, N$ ) have no terms of degree  $< c$ . In short, one can approximate the given formal solution by an analytic solution to arbitrarily high order. In particular, existence of an analytic solution will follow from that of a formal solution. The requirement  $\bar{y}(0) = 0$  is superfluous if the  $f_i(x, y)$  are polynomials in  $y$ .

Nagata's theorem cited above, to the effect that if  $\mathfrak{p}$  is a prime ideal in the convergent power series ring  $k\{x\} = k\{x_1, \dots, x_n\}$  then  $\mathfrak{p}k[[x]]$  is again prime, is an easy consequence of the approximation theorem. *Proof:* Let  $f_1(x), \dots, f_r(x)$  generate  $\mathfrak{p}$ , and suppose that  $\mathfrak{p}k[[x]]$  is not a prime ideal. Then there exist  $\bar{G}, \bar{H}, \bar{A}_i \in k[[x]]$  such that  $\bar{G}, \bar{H} \notin \mathfrak{p}k[[x]]$  and

$$\bar{G}\bar{H} = \sum_{i=1}^r \bar{A}_i f_i(x).$$

Applying the approximation theorem to the equation  $Y_{r+1}Y_{r+2} = \sum_{i=1}^r Y_i f_i(x)$ ,



we get  $G(x)$ ,  $H(x)$ ,  $A_i(x) \in k\{x\}$  which satisfy  $G \cdot H = \sum A_i f_i \in \mathfrak{p}$ . Moreover, as every ideal of a noetherian local ring is closed in the  $\mathfrak{m}$ -adic topology ( $\mathfrak{m}$  = the maximal ideal),  $G$  is not in  $\mathfrak{p}$  if it is sufficiently near  $\bar{G}$ . Similarly we may suppose that  $H$  is not in  $\mathfrak{p}$ . But  $GH$  is in  $\mathfrak{p}$ , contradiction.

The algebraic version of the approximation theorem runs as follows (Publ. I.H.E.S. 1970): *Let  $V$  be a variety (or a scheme of finite type) over a field  $k$ ; let  $\mathfrak{o}$  be the local ring of  $V$  at a point  $P$ . Let  $f(Y) = 0$  be a system of polynomial equations with coefficients in  $\mathfrak{o}$ :*

$$f = (f_1, \dots, f_r), \quad f_j(Y) \in \mathfrak{o}[Y_1, \dots, Y_N].$$

*Let  $\bar{y} = (\bar{y}_1, \dots, \bar{y}_N)$  be a solution of  $f(Y) = 0$  in  $\mathfrak{o}$ , and let  $c$  be an integer. Then there exist an étale neighborhood  $U$  of  $P$  in  $V$  and sections  $y_1, \dots, y_N$  of  $O_U$  such that  $f(y) = 0$  and  $y_j - \bar{y}_j \in \mathfrak{m}^c$ , where  $\bar{\mathfrak{m}}$  is the maximal ideal of  $\mathfrak{o}$ .*

One of the consequences of this theorem is the algebraizability of an isolated singular point of a complex analytic space. If  $P$  is an isolated singular point of a complex space  $V$  then a suitable neighborhood of  $P$  in  $V$  is isomorphic to an open set of an algebraic variety.

Artin has generalized the concept of scheme to that of algebraic space. A down-to-earth definition is the following: an algebraic space  $X$  consists of an affine scheme  $U$  (not necessary connected) and a closed subscheme  $R \subset U \times U$  such that (i)  $R$  is an equivalence relation and (ii) the projections  $p_i: R \rightarrow U$  ( $i=1, 2$ ) are étale. We view  $X$  as the quotient  $U/R$ . More intrinsically it is defined to be a contravariant functor  $X: (\text{Schemes})^\circ \rightarrow (\text{Sets})$  which is a sheaf for the étale topology of schemes, satisfying certain axioms which implies that  $X$  is a quotient sheaf  $U/R$  as above. Morphisms are defined so that algebraic spaces form a category. When  $U$  and  $R$  are schemes of finite type over  $\mathbb{C}$  the algebraic space  $X = U/R$  has a natural structure of complex space. It turns out that an irreducible compact complex space  $X$  of dimension  $n$  is an algebraic space if and only if it has  $n$

algebraically independent meromorphic functions. This shows that the concept of algebraic space is a natural one. Using the approximation theorem Artin has shown that some important functors of schemes are representable by algebraic spaces. One of the charms of algebraic geometry lies in that it allows attacks by several different methods. And, as we have seen, the methods of abstract algebraic geometry are not the weakest.

(end)

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