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Cosymplectic manifolds of constant φ-holomorphic curvature

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§0. Introduction.

Yano, K. and I. Mogi [4] already have proved that if a Kaehlerian manifold admits the axiom of holomorphic planes, then the manifold is of constant holomorphic curvature. Moreover, K. Ogiue has obtained a similar result to above theorem for a Sasakian manifold [3].

It is natural that we expect an analogous theorem to above result for a cosymplectic manifold. The purpose of the present paper is to prove that our expectation holds good.

§1. Characters of a cosymplectic manifold.

Let M be a $(2n+1)$-dimensional differentiable manifold covered by a system of coordinate neighborhoods $\{U, y^\lambda\}$, where here and in the sequel, the indices $\lambda, \mu, \nu, \kappa, \cdots\cdots$ run over the range $\{1, 2, \cdots\cdots, 2n+1\}$ and let M admits an almost contact structure, that is, a set (φ, ξ, η) of a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

(1. 1)
$$\varphi^2=-I+\eta\otimes\xi,$$
$$\eta\circ\varphi=0, \qquad \varphi\xi=0, \qquad \eta(\xi)=1.$$

We now assume that M admits a positive definite Riemannian metric G satisfying

(1. 2)
$$G(\varphi x, \varphi y)=G(x,y)-\eta(x)\eta(y),$$
$$\eta=G\xi, \qquad G(\xi, \xi)=1$$

and assume that x, y, z, and u are arbitrary vectors on M. In this case, we call such a differentiable manifold M an almost contact metric manifold.

An almost contact metric manifold M (φ, ξ, η, G) is said to be almost cosymplectic if the fundamental 2-form Φ defined by $\Phi(x,y)=G(\varphi x,y)$ and the 1-form η are both closed.

Moreover, an almost cosymplectic manifold M is said to be cosymplectic if M is normal, that is,

(1. 3) $$[\varphi,\ \varphi]+d\eta\otimes\xi=0,$$

where $[\varphi,\ \varphi]$ is the Nijenhuis tensor formed with φ.

Considering the normality (1. 3) and the closed properties of Φ and η, we can see that a cosymplectic manifold M is characterized by the following properties[2] :

(1. 4) $$'\nabla\varphi=0 \quad \text{and} \quad '\nabla\eta=0,$$

where $'\nabla$ means the operator of covariant differentiation with respect to the Christoffel symbol formed with G.

Differentiating the first equation of (1. 4) covariantly and using the Ricci formula, we have

(1. 5) $$v\circ(R(x,y)\varphi z)-v\circ\varphi(R(x,y)z)=0,$$

where R is the curvature tensor of M.

Similarly from the second equation of (1. 4), we get

(1. 6) $$\eta\circ R(x,y)=0.$$

Applying φ to (1. 5) and using above relation (1. 6), we obtain

(1. 7) $$G(R(x,y)\varphi z,\ \varphi u)=G(R(x,y)z,\ u)$$

and from which we find that the curvature tensor R of M is hybrid in the last two indices.

Moreover we have

(1. 8) $$G(R(x,y)\varphi z,\ u)=G(R(x,y)\varphi u,\ z)$$

and

(1. 9) $$G(R(\varphi x,y)z,\ u)=G(R(\varphi y,\ x)z,\ u)$$

by virtue of (1. 5).

Taking account of the relation (1. 7) and the Bianchi identity, we obtain additionally the following relation:

(1. 10) $$G(R(z,\ \varphi y)\varphi x,\ u)-G(R(z,\ \varphi x)\varphi y,\ u)=G(R(x,y)z,\ u).$$

§2. An example of a cosymplectic manifold.

In a $(2n+2)$-dimensional manifold, let K be a 2n-dimensional differentiable submanifold covered by a system of coordinate neighborhoods $\{V, x^h\}$, where here and in the sequel, the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots\cdots, 2n\}$ and let K admits a Kaehlerian structure, that is, a set (f, g) of a tensor field f of type $(1,1)$ and a positive definite Riemannian metric g satisfying

$$f^2=-I \quad \text{and} \quad g(fX, fY)=g(X, Y)$$

for arbitrary vector fields X and Y of K.

Putting

$$y^h=x^h \quad \text{and} \quad y^{2n+1}=\theta, \tag{2. 1}$$

we can construct a normal circle bundle $n(K)$ of K [1], [5] covered by a system of coordinate neighborhoods $\{U, y^\lambda\}$.

Introducing a covector l such that $\partial_i\theta=-l_i$ $(\partial_i=\partial/\partial x^i)$, we define a covector η of $n(K)$ by

$$\eta_i=l_i \quad \text{and} \quad \eta_{2n+1}=1$$

and a vector ξ of $n(K)$ by

$$\xi^h=0 \quad \text{and} \quad \xi^{2n+1}=1.$$

Constructing a structure φ and a Riemannian metric G by

$$\varphi=f^L, \qquad G=g^L+\eta\otimes\eta \tag{2. 2}$$

where (f, g) is the Kaehlerian structure of K and superscript index L means the horizontal lift with respect to (y^λ), we obtain [1], [5]

$$(\varphi_\mu{}^\lambda)=\begin{pmatrix} f_j{}^i & 0 \\ -f_j{}^t l_t & 0 \end{pmatrix}, \qquad (G_{\mu\lambda})=\begin{pmatrix} g_{ji}+l_j l_i & l_j \\ l_i & 1 \end{pmatrix}. \tag{2. 3}$$

Therefore our normal circle bundle $n(K)$ admits an almost contact metric structure (φ, ξ, η, G).

It is easily seen that the metric tensor G has the contravariant components

$$(G^{\mu\lambda})=\begin{pmatrix} g^{ji} & -l^i \\ -l^j & 1+l_t l^t \end{pmatrix}, \tag{2. 4}$$

and the Christoffel symbols $\Gamma_\mu{}^\nu{}_\lambda$ formed with G are given by the relations:

(*) $\Gamma_j{}^k{}_i=\{{}_j{}^k{}_i\}$ and all other components of Γ vanish, where $\{{}_j{}^k{}_i\}$ are the Christoffel symbols formed with g.

Since the vecsor $B_i{}^\lambda=\partial_i y^\lambda$ $(\partial_i=\partial/\partial x^i)$ for each i has components

$$(2.5)\qquad (B_i{}^\lambda)=\begin{pmatrix}\delta_i{}^k\\ -l_i\end{pmatrix},$$

and the vector ξ is normal to K, we can see that K is an invariant hypersurface of $n(K)$. Therefore we obtain the relation

$$(2.6)\qquad \nabla_X B(Y)=h(X,Y)\xi,$$

where h is the second fundamental tensor of K, X and Y are arbitrary vectors of K and ∇ means the operator of covariant differentiation with respect to $\{{}_j{}^k{}_i\}$.

Taking account of the $2n+1$-th contravariant component of the equation (2.6), we find that $h_{ji}=-\nabla_j l_i$ and from which we obtain

$$(1)\qquad d\eta=0$$

because of the symmetric property of the tensor h.

Defining $'\Omega_{\nu\mu\lambda}=3'\nabla_{[\nu}\varphi_{\mu\lambda]}$ and $\Omega_{kji}=3\nabla_{[k}f_{ji]}$, we have $0=\Omega_{kji}={}'\Omega_{\nu\mu\lambda}B_k{}^\nu B_j{}^\mu B_i{}^\lambda$ and $'\Omega_{kji}=\Omega_{kji}=0$, $'\Omega_{\nu\mu\ 2n+1}=0$, where $'\nabla$ is the operator we have in (1.4). Therefore we obtain

$$(2)\qquad d\Phi=0$$

where Φ is the fundamental 2-form of $n(K)$ defined by $\Phi(x,y)=G(\varphi x,y)$ for arbitrary vector fields x and y. Thus $n(K)$ is an almost cosymplectic manifold.

Computing the components of the tensor S of type (1,2) defined by

$$(2.7)\qquad S(x,y)=[\varphi x,\varphi y]-\varphi[\varphi x,y]-\varphi[x,\varphi y]+\varphi^2[x,y]+d\eta(x,y)\xi$$

for arbitrary two vector fields x and y of $n(K)$, we can see easily that

$$(3)\qquad S_\nu{}^\lambda{}_\mu=0$$

since the Nijenhuis tensor of K vanishes and $\nabla_X l(Y)=0$.

Taking account of the relations (1), (2) and (3), we conclude that $n(K)$ is a cosymplectic manifold. Thus we constructed an example of a cosymplectic manifold.

§ 3. Cosymplectic manifolds of constant φ-holomorphic curvature.

In a cosymplectic manifold M, we call a sectional curvature

$$k=-\frac{G(R(\varphi x, x)\varphi x, x)}{G(x, x)G(\varphi x, \varphi x)} \tag{3.1}$$

determined by two orthogonal vectors x and φx the φ-holomorphic sectional curvature with respect to the vector x of M. If the φ-holomorphic sectional curvature is always constant with respect to any vector at every point of the manifold M then we call the manifold M a manifold of constant φ-holomorphic curvature.

Now, if this is the case, then (3.1) or

$$\begin{aligned} &G(R(\varphi x, y)\varphi z, u)+G(R(\varphi y, z)\varphi x, u)+G(R(\varphi z, x)\varphi y, u) \\ &=-k[G(x, y)G(z, u)+G(x, z)G(y, u)+G(y, z)G(x, u) \\ &-G(x, y)\eta(z)\eta(u)-G(x, z)\eta(y)\eta(u)-G(y, z)\eta(x)\eta(u)] \end{aligned} \tag{3.2}$$

should be satisfied for arbitrary vectors x, y, z and u of M.

Replacing φx and φz with x and z respectively and taking account of (1.1), (1.2), (1.6), (1.7) and (1.8), we obtain

$$\begin{aligned} &G(R(x, y)z, u)-G(R(y, z)x, u)-G(R(z, \varphi x)\varphi y, u) \\ &=-k[G(\varphi x, y)G(\varphi z, u)+G(x, z)G(y, u)-\eta(x)\eta(z)G(y, u) \\ &+G(\varphi x, u)G(\varphi z, y)-G(x, z)\eta(y)\eta(u)+\eta(x)\eta(y)\eta(z)\eta(u)]. \end{aligned} \tag{3.3}$$

Denoting the equation which is obtained by replacing x with y on (3.3) by (3.3)$'$ and subtracting (3.3)$'$ from the equation (3.3), we obtain

$$\begin{aligned} G(R(x, y)z, u)=-\frac{k}{4}[&G(x, z)G(y, u)-G(y, z)G(x, u) \\ &+G(\varphi x, z)G(\varphi y, u)-G(\varphi y, z)G(\varphi x, u)+2G(\varphi x, y)G(\varphi z, u) \\ &-\eta(x)\eta(z)G(y, u)+\eta(y)\eta(z)G(x, u) \\ &+\eta(x)\eta(u)G(y, z)-\eta(y)\eta(u)G(x, z)]. \end{aligned} \tag{3.4}$$

It is easily to be seen from the Bianchi identity that, if the curvature tensor has the form (3.4), the scalar curvature k is an absolute constant. Summarizing above results, we have the following:

THEOREM 3.1. *If a cosymplectic manifold has a constant φ-holomorphic sectional curvature at every point, then the curvature tensor R of the manifold* *is of the form* (3. 4), *where k is a constant.*

We now assume that, when there is given a φ-holomorphic plane element, that is, a plane element determened by the vectors φx and $\varphi^2 x$ at a point of the manifold, where x is a unit vector and not parallel to ξ, we can always draw a 2-dimensional totally geodesic surface passing through this point and being tangent to the given φ-holomorphic plane element. If this is the case, we say that the manifold satisfies the axiom of φ-holomorphic planes. In this case, it is well known that the curvature tensor R has components

$$G(R(\varphi x, \varphi^2 x)\varphi^2 x, y) = A \cdot G(\varphi^2 x, y) + B \cdot G(\varphi x, y). \tag{3. 5}$$

Applying (1. 1), replacing y with x and taking account of $G(x, \xi) \neq \pm 1$ since x is not parallel to ξ, we find that $A=0$. Thus (3. 5) becomes

$$G(R(\varphi x, x)x, y) - BG(\varphi x, y)G(x, x) = 0. \tag{3. 6}$$

Putting

$$k = \frac{B}{\|x\|} = \frac{BG(x, x)}{G(x, x)\eta(x)\eta(x)},$$

we obtain

$$G(R(\varphi x, x)x, y) - k[G(x, x) - \eta(x)\eta(x)]G(\varphi x, y) = 0, \tag{3. 7}$$

by virtue of (3. 6).

Since the equation (3. 7) is satisfied for arbitrary vector fields x and y, we have

$$\begin{aligned} &G(R(\varphi z, y)x, u) + G(R(\varphi x, z)y, u) + G(R(\varphi y, x)z, u) \\ &= k[G(\varphi x, u)G(y, z) + G(\varphi y, u)G(x, z) + G(\varphi z, u)G(x, y) \\ &\quad - G(\varphi x, u)\eta(y)\eta(z) - G(\varphi z, u)\eta(x)\eta(y) - G(\varphi y, u)\eta(z)\eta(x)]. \end{aligned} \tag{3. 8}$$

Replacing z with φz on (3. 8), we get

$$\begin{aligned} &-G(R(z, y)x, u) + G(R(x, z)y, u) + G(R(\varphi y, x)\varphi z, u) \\ &= k[G(\varphi x, u)G(\varphi z, y) + G(\varphi y, u)G(\varphi z, x) - G(y, x)G(z, u) \\ &\quad + G(y, x)\eta(z)G(\xi, u) + \eta(y)\eta(x)G(z, u) - \eta(z)\eta(x)\eta(y)G(\xi, u)]. \end{aligned}$$

Denoting the equation which is obtained by replacing z with y on above equation by (3. 9)′ and subtracting (3. 9)′ from (3. 9), we obtain finally

$$G(R(z, y)x, u) = -\frac{k}{4}[G(z, x)G(y, u) - G(y, x)G(z, u) + G(\varphi z, x)G(\varphi y, u)$$

$$-G(\varphi y, x)G(\varphi z, u)+2G(\varphi z, y)G(\varphi x, u)+G(y, x)\eta(u)\eta(z)$$
$$-G(x, z)\eta(u)\eta(y)+G(z, u)\eta(y)\eta(x)-G(y, u)\eta(z)\eta(x)],$$

which shows that the manifold is of constant φ-holomorphic curvature. Thus we have proved the following:

THEOREM 3. 2. *If a cosymplectic manifold admits the axiom of φ-holomorphic planes, then the manifold is of constant φ-holomorphic curvature.*

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