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A NATURAL APPROACH TO FREDHOLM STRUCTURES ON BANACH MANIFOLDS

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Introduction. The author studies in this paper to link the Fredholm structures of Banach spaces and differential structures on Banach manifolds more inclusively. The approaching should be a natural way and abstracted in the following three parts.

Although some elementary analytical approach to the Fredholm structures of Banach spaces is discussed in [1] with detail self-contained direct method, and some general discussion of it is found in [6], in Part A, however, is used the notion of the principal fibre bundle [3] to produce the same Fredholm structures, and the author thinks that it is more natural.

In Part B the author defines Φ structures that cover all smooth Banach manifolds, and reduces them to $\mathcal{C}$-structures as a particular case.

The theorem in Part C gives a characterization of $\mathcal{C}(M, E)$, the set of all $\mathcal{C}$-admissible maps of $\mathcal{C}$-manifold M into E, the Banach space which models M. The argument of this part should refer to [5] heavily.

A. Let X be a topological space and G a topological group acting continuously on the right of X, i.e., the right action $X\times G\to X$, defined by $(x,g)\to x\cdot g$ such that $x\cdot(g_1g_2)=(x\cdot g_1)\cdot g_2$ and $x\cdot 1=x$ for all x in X and g_1, g_2 in G, where 1 is the identity in G, is continuous.

A *G-bundle* or a ***principal fibre bundle with the structural group*** G is the surjective map $p: X\to B$ of X onto the orbit space $B=X/G$ with the quotient topology which assigns to each x in X its orbit $p(x)\in B$ under the G-action such that the G-action is *principal* which means that: (1) The G-action is free, i.e., $x\cdot g=x$ only when $g=1$, (2) it is proper, i.e., the map $\theta: \Delta\to G$ defined by $\theta(x, x\cdot g)=g$ is continuous, where $\Delta=\{(x,x')\in X\times X \mid x'=x\cdot g$ for some (unique) $g\in G\}$ and (3) it is locally trivial, i.e., each orbit b in B has a neighborhood on which there exists a continuous section. We call the acting group G the *structural group* of the G-bundle.

We denote by E an infinite dimensional Banach space, $L(E)$ the Banach space of all continuous linear maps of E into itself with the Sup. norm, and $C(E)$ the closed subspace of $L(E)$ consisting of compact operators [1]. If we think of $L(E)$ as a Banach Lie algebra with unit I, the identity operator, and bracket $[T,S]=T\cdot S-S\cdot T$ for $T,S\in L(E)$, then $C(E)$ is a bilateral ideal without unit of $L(E)$. Let the canonical homomorphism $p: L(E)\to L(E)/C(E)$ be the topological identification. We shall find the structural group G with which p becomes a principal fibre bundle.

THEOREM. *The structural group G that makes $p: L(E)\to L(E)/C(E)$ into a G-bundle is contained in the subset $\{I+k \mid k\in C(E)\}$ of $L(E)$.*

Proof. Since G must act on the right of $L(E)$ continuously, the map $L(E)\times G\to L(E)$

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defined by $(T, g) \longrightarrow T \cdot g$ is continuous, and the orbit space $L(E)/G$ of the group G must be equal to $L(E)/C(E)$.

For any T in $L(E)$ the orbit $p(T)$ of T under the G-action must be expressed by $p(T) = \{T \cdot g \mid g \in G\}$. On the other hand, by the canonical homomorphism $p, p(T) = T + C(E)$. Thus $T + C(E) = \{T \cdot g \mid g \in G\}$. Hence for any g in G there exists a k in $C(E)$ such that $T + k = T \cdot g$ and this equation holds if the binary operation is considered as the composition and g as $I + k'$ for some k' in $C(E)$. Therefore g belongs to the set $\{I + k \mid k \in C(E)\}$. This proves that the structural group G is contained in the set $\{I + k \mid k \in C(E)\}$ and G contains I as the identity, since $1 = I + 0 = I$ and $0 \in C(E)$.

We can easily see that with this $G \subset \{I + k \mid k \in C(E)\}$ the G-action is continuous, free, proper and locally trivial by the routine checks, which implies that $p: L(E) \to L(E)/C(E)$ is a G-bundle.

The structural group G found in the above theorem within $L_C(E) = \{I + k \mid k \in C(E)\}$ is called the Fredholm group of E [4] and denoted by $GL_C(E)$ [5]. The author contends that the above theorem has produced the group $GL_C(E)$ by a natural way.

Denote $GL(E) = \{T \in L(E) \mid T \text{ has its inverse}\}$. Then $GL(E)$ is an open group (under composition) in $L(E)$ [1] and $GL_C(E)$ is an invariant subgroup of $GL(E)$, and hence $\pi: GL(E) \to GL(E)/GL_C(E)$ is a $GL_C(E)$-bundle as the canonical homomorphism.

On the other hand denote $\Phi(E) = \{T \in L(E) \mid \dim \ker T,\ \dim \operatorname{coker} T < \infty\}$ and an element of $\Phi(E)$ is called a *Fredholm operator* of E. Defining index of T by ind $T = \dim \ker T - \dim \operatorname{coker} T$, we have the map ind$: \Phi(E) \to Z$ of $\Phi(E)$ into integers set Z and denote $\Phi_0(E) = \text{ind}^{-1}(0)$. If we denote $G_0 = p(\Phi_0(E))$, then $p: \Phi_0(E) \to G_0$ is a $GL_C(E)$-bundle as the subbundle of $p: L(E) \to L(E)/C(E)$ induced by the restriction. G_0 is a group isomorphic with $GL(E)/GL_C(E)$ and the elementary detail proof of it is given in [1]. With this isomorphism η we have the commutative diagram

$$\begin{tikzcd} GL(E) \arrow[r, "i"] \arrow[d, "\pi"'] & \Phi_c(E) \arrow[d, "p"] \\ GL(E)/GL(E) \arrow[r, "\eta"] & G_0 \end{tikzcd}$$

where i is the inclusion. We call $G_0 = GL(E)/GL_C(E)$ the *Fredholm structure of E*. We notice that $p: \Phi_0(E) \to G_0$ is a homotopy equivalence, since its fibres are contractible.

B. We will denote by M a smooth (infinite differentiable) manifold modelled on E, denoted by E-manifold. We define that M *admits a Φ-structure* if there is a collection Φ_M of charts (U_i, Q_i) covering M and satisfying that for all i, j and $x \in U_i \cap U_j$,

$$D(\theta_j \theta_i^{-1})(\theta_i(x)) \in \Phi(E),$$

where D denotes the differentiation operator. A member of Φ_M is often called Φ-chart. If (V, φ) is a differentiable chart of M such that for each $x \in V$, there exists a Φ-chart (U_i, θ_i) so that $D(\varphi\theta_i^{-1})(\theta_i(x)) \in \Phi(E)$, then $D(\varphi\theta_j^{-1})(\theta_j(x)) = D(\varphi\theta_i^{-1}) \circ D(\theta_i\theta_j^{-1})(\theta_j(x)) \in \Phi(E)$ for all j with $U_i \cap U_j \neq \Phi$ and $x \in U_i \cap U_j$. Therefore we can always construct a maximal collection Φ_M which is meant by Φ-structure admitted on M, and M with Φ_M is called a Fredholm manifold, denoted by Φ-manifold.

We will apply the similar notations from $\Phi(E)$ to $GL_C(B)$ by using $\mathcal{C}_M$ and $\mathcal{C}$-manifold, etc..

THEOREM. *Every smooth manifold modelled on any Banach space is a Φ-manifold.*

Proof. If we recall the definition of the tangent space TM_x of M at x, a tangent vector at $x \in M$ is an equivalence class of all triples (U_i, θ_i, v), where the equivalence relation $(U_i, \theta_i, v) \sim (U_j, \theta_j, w)$ is given by the condition $D(\theta_j\theta_i^{-1})(\theta_i(x))v = w$ for $v \in E_i$, $w \in E_j$ (with $E_i = E_j = E$). We can easily see that $D(\theta_j\theta_i^{-1})(\theta_i(x)): E_i \to E_j$ is a toplinear isomorphism [2]. Hence $D(\theta_j\theta_i^{-1})(\theta_i(x)$ has its inverse $[D(\theta_j\theta_i^{-1})]^{-1}(\theta_j(x)) = D(\theta_i\theta_j^{-1})(\theta_j(x))$, and so in fact $(D(\theta_j\theta_i^{-1})(\theta_i(x)) \in GL(E) \subset \Phi_0(E) \subset \Phi(E)$ for all i, j and $x \in U_i \cap U_j$.

COROLLARY. *Any coordinate transformation $\theta_j\theta_i^{-1}$ on $U_i \cap U_j \subset M$ has its derivative at $x \in U_i \cap U_j$ expressible as*

$$D(\theta_j\theta_i^{-1})(\theta_i(x)) = T + k,$$

where $T \in GL(E)$ and $k \in C(E)$.

Proof. Since $D(\theta_j\theta_i^{-1})(\theta_i(x)) \in GL(E)$, there exists some T in $GL(E)$ such that $D(\theta_j\theta_i^{-1})(\theta_i(x))$ is contained in the orbit of T under the Fredholm group $GL_C(E)$, i.e., contained in $T \cdot GL_C(E) \subset T + C(E)$.

Due to the last corollary, $\mathcal{C}$-manifold is the manifold M that admits Fredholm structure $\mathcal{C}_M$ such that all coordinate transfomations have their derivatives contained in the orbit of $I \in GL(E)$ under the action group $GL_C(E)$. We can therefore give the following remark as a consequence.

REMARK The tangent bundle $\pi: TM \to M$ is a $GL(E)$-bundle for any smooth E-manifold, and in particular it is a $GL_C(E)$-bundle for any $\mathcal{C}$-manifold modelled on E.

C. Let M and N be smooth manifolds modelled on Banach spaces E and F respectively, and $f: M \to N$ be a smooth map. f is called a *Fredholm map* if its differential $f_*(x) = Df_x$ at each x in M is a Fredholm operator of TM_x into $TN_{f(x)}$, and denoted by Φ-map. We define the index of f by that of $f_*(x)$ when M is connected. We therefore assume that M is connected whenever this notion is used.

One of nice visible non-trivial examples of Φ-map the index of which is purposely pursued is found in [7] with an infinite dimensional Hilbert manifold.

Now let M and N be $\mathcal{C}$-manifolds modelled on the same Banach space E. A map $f: M \to N$ is said to be *$\mathcal{C}$-admissible* if the vector bundle map $Df: TM \to TN$ has the form

$$Df(x, v) = (f(x), v + k(x)v)$$

with $k(x) \in C(E)$ for each x in M and v in TM_x, where TM and TN are given the $\mathcal{C}$-structure induced from M and N respectively.

If we refer to [6], the index function $\mathrm{ind}: \Phi(E) \to Z$ has the property that $\mathrm{ind}(T+k) = \mathrm{ind}\, T$ for any T in $\Phi(E)$ and k in $C(E)$. Hence $Df_x = I + k(x): TM_x \to TN_{f(x)}$ has index zero at each x in M. Therefore we have the following:

REMARK. Any $\mathcal{C}$-admissible map is a Φ_0-map, a Fredholm map of index zero.

The following lemma is a particular case of so-called "the pull-back $\mathcal{C}$-structure $\{M, N, f\}_{\mathcal{C}}$ on M" of a Φ_0-map $f: M \to N$, where M and N are $\mathcal{C}$-manifolds modelled on F and E respectively (Theorem 2.2 in [5]). In the sequel E is understood to have the canonical $\mathcal{C}$-structure.

LEMMA. *If M is a $\mathcal{C}$-manifold with $\mathcal{C}_M$ modelled on E and admits a smooth partitions* *of unity, then there is a Φ_0-map $f:M\to E$ such that f is $\mathcal{C}$-admissible.*

Proof. Let $\{U_i,\ \theta_i)\}$ be the coordinate covering of M by admissible charts and $\{\eta_i\}$ be the partitions of unity subordinate to $\{U_i\}$. Define $f:M\to E$ by $f(x)=\sum_i\eta_i(x)\theta_i(x)$. Then for any chart (U,θ) in $\mathcal{C}_M$ with $\theta(x)=0$,

$$D(f\theta^{-1})_0 v=\textstyle\sum_i\eta_i(x)D(\theta_i\theta^{-1})_0 v+\sum_i(D(\eta_i\theta^{-1})_0\theta_i(x))v$$

and the maps of the first and second terms belong to $L_C(E)$ and $C(E)$ respectively. Therefore we can see that f is $\mathcal{C}$-admissible if we change the right side into the form $I+k(x)$ as a consequence.

Denote by $\mathcal{C}(M,E)$ the set of all $\mathcal{C}$-admissible maps of M into E. Then by this lemma it is non-empty, and by that remark it is a subset of $\Phi_0(M,E)$, the set of all Φ_0-maps of M into E.

For any pair f,g in $\Phi_0(M,E)$, f and g are said to be *Φ_0-homotopic* if there exists a homotopy $h:M\times[0,1]\to E$ such that for each t in $[0,1]$, h_t: $M\to E$ are Φ_0-maps with $h_0=f$ and $h_1=g$, called Φ_0-homotopy. We denote by $\Phi_0[M,E]$ the Φ_0-homotopic equivalence classes in $\Phi_0(M,E)$.

Denoting by $[M,\Phi_0(E)]$ the homotopy classes of all continuous mappings of M into $\Phi_0(E)$, we have the bijection ρ: $\Phi_0[M,E]\to[M,\Phi_0(E)]$ defined by $\rho[f]=[Df\circ\tau]$, where $(Df\circ\tau)(x)=Df_x\circ\tau_x$, $[\ \ ]$ denotes the equivalence class and τ the trivialization τ: $M\times E\to TM$ after a group structure is induced in $[M,\Phi_0(E)]$ by the $GL_C(E)$-bundle p: $\Phi_0(E)\to G_0$ (due to the Proposition 2.4 in [5]).

THEOREM. *$\mathcal{C}(M,E)$ is exactly the same class as $[f]$, where f is the Φ_0-map of M into E constructed in the proof of the preceding lemma.*

Proof. Let. $f\in\mathcal{C}(M,E)$ be the particular map constructed in the proof of the preceding lemma. It is clear that $[f]$ is mapped onto the identity of $[M,\Phi_0(E)]$ by ρ and also it is obvious that for any $g\in\mathcal{C}(M,E)$, $Df_x\circ\tau_x-Dg_x\circ\tau_x\in C(E)$. Hence $p\circ Df_x\circ\tau_x=p\circ Dg_x\circ\tau_x$ in G_0. Since p: $\Phi_0(E)\to G_0$ is the homotopy equivalence, $[Df\circ\tau]=[Dg\circ\tau]$. Now by the bijection ρ, $[g]=[f]$, i.e., $g\in[f]$. which implies that $\mathcal{C}(M,E)\subset[f]$.

Conversely if $g\in[f]$, $[Dg\circ\tau]$ is the identity of $[M,\Phi_0(E)]$ and so $(p\circ Dg\circ\tau)(x)=I.\ GL_C(E)\in GL(E)/GL_C(E)$ by the commutative diagram in Part A, i.e., $Dg_x\circ\tau_x=I+k(x)$, where $k:M\to C(E)$, which implies that $[f]\subset\mathcal{C}(M,E)$. This completes the proof.

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