J. Korean Math. Soc.
Vol. 9, No. 2, 1972

INVARIANT SUBMANIFOLDS OF A MANIFOLD WITH QUASI-NORMAL (f, g, u, v, λ)-STRUCTURE

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Introduction

Yano and Okumura [3] have recently introduced the so-called (f, g, u, v, λ)-structure in an even-dimensional manifold and studied invariant submanifolds of a manifold with normal (f, g, u, v, λ)-structure [4], [5].

Kubo [1] also studied invariant submanifolds of codimension 2 of a manifold with (f, g, u, v, λ)-structure.

The purpose of the present paper is to study invariant submanifolds of a manifold with quasi-normal (f, g, u, v, λ)-structure.

We state in §1 some known results for an (f, g, u, v, λ)-structure and recall invariant submanifolds of a manifold with such structure.

In §2 and 3, we study odd-dimensional invariant submanifolds of a manifold with quasi-normal (f, g, u, v, λ)-structure.

§1. Invariant submanifolds of a manifold with (f, g, u, v, λ)-structure.

Let M be a differentiable manifold with an (f, g, u, v, λ)-structure, that is, a differentiable manifold endowed with a tensor field f of type $(1,1)$, a Riemannian metric g, two 1-forms u and v and a function λ satisfying

$$
\begin{aligned}
&f_j{}^t f_t{}^h = -\delta_j^h + u_j u^h + v_j v^h,\\
&f_j{}^s f_i{}^t g_{st} = g_{ji} - u_j u_i - v_j v_i,\\
&u_t f_j{}^t = \lambda v_j, \quad f_t{}^h u^t = -\lambda v^h,\\
&v_t f_j{}^t = -\lambda u_j, \quad f_t{}^h v^t = \lambda u^h,\\
&u_t u^t = v_t v^t = 1-\lambda^2, \quad u_t v^t = 0,
\end{aligned}
\tag{1.1}
$$

$f_i{}^h$, g_{ji}, u_i, v_i and λ being respectively components of f, g, u, v and λ with respect to a local coordinate system, u^h and v^h being defined by

$$u_i = g_{ih} u^h \quad \text{and} \quad v_i = g_{ih} v^h$$

respectively, where here and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, 2m\}$. It is known that such a manifold is even-dimensional.

If we put $f_{ji} = f_j{}^t g_{ti}$, we can easily see that f_{ji} is skew-symmetric.

We put

$$S_{ji}{}^h = N_{ji}{}^h + (\nabla_j u_i - \nabla_i u_j) u^h + (\nabla_j v_i - \nabla_i v_j) v^h, \tag{1.2}$$

$N_{ji}{}^h$ denoting the Nijenhuis tensor formed with $f_i{}^h$ and ∇_i the operator of covariant differentiation with respect to Christoffel symbols $\{{}_j{}^h{}_i\}$ formed with g_{ji}. If $S_{ji}{}^h$ vanish-

Received by the editors Oct. 23, 1972.

es, we say that the (f, g, u, v, λ)-structure is *normal*.

The (f, g, u, v, λ)-structure is said to be *quasi-normal* if it satisfies

(1.3) $$T_{jih}=S_{jih}-(f_j{}^t f_{tih}-f_i{}^t f_{tjh})=0,$$

where $S_{jih}=g_{th}S_{ji}{}^t$, $f_{jih}=\nabla_j f_{ih}+\nabla_i f_{hj}+\nabla_h f_{ji}$.

Yano and one of the present authors proved the following two theorems [2]:

THEOREM 1.1. *In a manifold with quasi-normal (f, g, u, v, λ)-structure, we have*

(1.4) $$f_j{}^t\nabla_h f_{ti}-f_i{}^t\nabla_h f_{tj}=u_j\nabla_i u_h-u_i\nabla_j u_h+v_j\nabla_i v_h-v_i\nabla_j v_h.$$

THEOREM 1.2. *Let M be a complete manifold with normal (or quasi-normal) (f, g, u, v, λ)-structure satisfying*

$$\nabla_j v_i-\nabla_i v_j=2cf_{ji}$$

or equivalently

$$\nabla_j u_i+\nabla_i u_j=-2c\lambda g_{ji},$$

c being non-zero constant. If the function $\lambda(1-\lambda^2)$ does not vanish almost everywhere and dim $M>2$, *then M is isometric with an even-dimensional sphere.*

We consider a submanifold N of M represented by $x^h=x^h(y^a)$ and put $B_b{}^h=\partial_b x^h$, $\partial_b=\partial/\partial y^b$, where here and throughout the paper the indices a, b, c, d, e run over the range $\{1, 2, \cdots, n\}$, $n<2m$.

The induced Riemannian metric is given by

(1.5) $$g_{cb}=g_{ji}B_c{}^jB_b{}^i.$$

We denote by $C_x{}^h$ $2m-n$ mutually orthogonal unit normals to N. Then equations of Gauss and those of Weingarten are respectively

(1.6) $$\nabla_c B_b{}^h=\sum_x h_{cbx}C_x{}^h$$

and

(1.7) $$\nabla_c C_x{}^h=-h_c{}^a{}_x B_a{}^h+\sum_y l_{cxy}C_y{}^h,$$

where

(1.8) $$\nabla_c B_b{}^h=\partial_c B_b{}^h+\left\{{h \atop j\,i}\right\}B_c{}^jB_b{}^i-\left\{{a \atop c\,b}\right\}B_a{}^h$$

is the Van der Waerden-Borotolotti covariant differentiation of $B_b{}^h$, $\left\{{a \atop c\,b}\right\}$ being Christoffel symbols formed with g_{cb},

(1.9) $$\nabla_c C_x{}^h=\partial_c C_x{}^h+\left\{{h \atop j\,i}\right\}B_c{}^jC_x{}^i,$$

h_{cbx} components of second fundamental tensors with respect to normals $C_x{}^h$, $h_c{}^a{}_x=h_{cbx}g^{ba}$ and l_{cxy} components of the third fundamental tensor with respect to normals $C_x{}^h$.

We assume that the submanifold N of M is f-invariant, that is, the transform of a vector tangent to N by the linear transformation f is always tangent to N:

(1.10) $$f_i{}^hB_b{}^i=f_b{}^aB_a{}^h,$$

$f_b{}^a$ being a tensor field of type (1, 1) of N. This shows that

(1.11) $$f_{ih}B_b{}^iC_x{}^h=0.$$

Thus, we put

(1.12) $$f_i{}^hC_x{}^i=\sum_y r_{xy}C_y{}^h,$$

from which,

(1.13) $$r_{xy}=-r_{yx}.$$

We put

(1.14) $$u^h=B_a{}^h u^a+\sum_x \alpha_x C_x{}^h,$$

(1.15) $$v^h=B_a{}^h v^a+\sum_x \beta_x C_x{}^h,$$

u^a and v^a being vector fields of N, α_x and β_x being functions of N.

From (1.1), (1.10), (1.12), (1.14) and (1.15), we find

(1.16) $$f_b{}^c f_c{}^a=-\delta_b^a+u_b u^a+v_b v^a,$$

(1.17) $$f_c{}^e f_b{}^d g_{ed}=g_{cb}-u_c u_b-v_c v_b,$$

(1.18) $$f_b{}^a u^b=-\lambda v^a, \qquad f_b{}^a v^b=\lambda u^a,$$

(1.19) $$u_a u^a=1-\lambda^2-\sum_x \alpha_x{}^2, \qquad v_a v^a=1-\lambda^2-\sum_x \beta_x{}^2,$$

(1.20) $$u_a v^a=-\sum_x \alpha_x \beta_x,$$

(1.21) $$\alpha_x u_b+\beta_x v_b=0,$$

(1.22) $$\sum_y r_{xy} r_{yz}=-\delta_{xz}+\alpha_x \alpha_z+\beta_x \beta_z,$$

(1.23) $$\sum_x r_{xy} \alpha_x=-\lambda \beta_y, \quad \sum_x r_{xy} \beta_x=\lambda \alpha_y.$$

We also have from (1.10), $f_{ji} B_c{}^j B_b{}^i=f_c{}^e g_{eb}$. Thus putting $f_c{}^e g_{eb}=f_{cb}$, we see that f_{cb} is skew-symmetric.

It will be easily verified that, for invariant submanifold N, we have

(1.24) $$N_{ji}{}^h B_c{}^j B_b{}^i=[f, f]_{cb}{}^a B_a{}^h,$$

$[f, f]_{cb}{}^a$ being the Nijenhuis tensor formed with $f_b{}^a$.

Since

$$(\nabla_j u_i-\nabla_i u_j) B_c{}^j B_b{}^i=\nabla_c (u_i B_b{}^i)-u_i \nabla_c B_b{}^i-\nabla_b (u_j B_c{}^j)+u_j \nabla_b B_c{}^j,$$

that is,

$$(\nabla_j u_i-\nabla_i u_j) B_c{}^j B_b{}^i=\nabla_c u_b-\nabla_b u_c,$$

from which, using (1.14),

(1.25) $$(\nabla_j u_i-\nabla_i u_j) u^h B_c{}^j B_b{}^i=(\nabla_c u_b-\nabla_b u_c) u^a B_a{}^h+\sum_x \alpha_x (\nabla_c u_b-\nabla_b u_c) C_x{}^h.$$

Similarly we can prove that

(1.26) $$(\nabla_j v_i-\nabla_i v_j) v^h B_c{}^j B_b{}^i=(\nabla_c v_b-\nabla_b v_c) v^a B_a{}^h+\sum_x \beta_x (\nabla_c v_b-\nabla_b v_c) C_x{}^h.$$

On the other hand, denoting $\nabla^h=g^{th}\nabla_t$ and $f_{ji}{}^h=g^{th} f_{jit}$, we can write

$$f_{ti}{}^h=\nabla_t f_i{}^h-\nabla_i f_t{}^h+\nabla^h f_{ti},$$

from which, using (1.10) and (1.11),

$$f_j{}^t f_{ti}{}^h B_c{}^j B_b{}^i=f_c{}^a (\nabla_t f_i{}^h-\nabla_i f_t{}^h+\nabla^h f_{ti}) B_a{}^t B_b{}^i=f_c{}^a \{\nabla_a (f_b{}^e B_e{}^h)-\nabla_b (f_a{}^e B_e{}^h)+\nabla^h f_{ab}\}$$

because of $\nabla_b B_c{}^h=\nabla_c B_b{}^h$.

If we take account of (1.6) and (1.16), then the last equation becomes

(1.27) $$f_j{}^t f_{ti}{}^h B_c{}^j B_b{}^i=f_c{}^e f_{eb}{}^a B_a{}^h+f_c{}^a f_b{}^e \sum_x h_{aex} C_x{}^h-(-\delta_c^e+u_c u^e+v_c v^e) \sum_x h_{bex} C_x{}^h,$$

where $f_{bc}{}^a=g^{ea} f_{bce}$, $f_{ebc}=\nabla_e f_{bc}+\nabla_b f_{ce}+\nabla_c f_{eb}$.

Taking the skew-symmetric part of (1.27) in indices b and c, we find

(1.28) $$(f_j^t f_{ti}^h - f_i^t f_{tj}^h) B_c^j B_b^i = (f_c^e f_{eb}^a - f_b^e f_{ec}^a) B_a^h + \sum_x \{(u_b u^e + v_b v^e) h_{cex} - (u_c u^e + v_c v^e) h_{bex}\} C_x^h.$$

From (1.24), (1.25), (1.26) and (1.28), we have

(1.29) $$\{S_{ji}^h - (f_j^t f_{ti}^h - f_i^t f_{tj}^h)\} B_c^j B_b^i = \{[f,f]_{cb}^a + (\nabla_c u_b - \nabla_b u_c) u^a + (\nabla_c v_b - \nabla_b v_c) v^a - (f_c^e f_{eb}^a - f_b^e f_{ec}^a)\} B_a^h + \sum_x \{(\nabla_c u_b - \nabla_b u_c)\alpha_x + (\nabla_c v_b - \nabla_b v_c)\beta_x + h_{cex}(u_b u^e + v_b v^e) - h_{bex}(u_c u^e + v_c v^e)\} C_x^h.$$

It is known that [5]

THEOREM 1.3. *Let N be an invariant submanifold of a manifold with (f, g, u, v, λ)-structure. If there exists a point P of N such that λ does not vanish at P, then the submanifold N is even-dimensional. If λ vanishes identically on N, then N is odd-dimensional.*

Equations (1.16)-(1.20) show that a necessary and sufficient condition f_b^a, g_{cb}, u_b, v_b and λ to define an (f, g, u, v, λ)-structure is that $\alpha_x=0$, $\beta_x=0$, that is, the vectors u^h and v^h are always tangent to the submanifold N.

Taking account of (1.29) and Theorem 1.3, we have

PROPOSITION 1.4. *Let M be a differentiable manifold with quasi-normal (f, g, u, v, λ)-structure such that $\lambda \neq 0$ almost every where along N and u^h* and *v^h are always tangent to N. Then, the even-dimensional submanifold N admits also a quasi-normal (f, g, u, v, λ)-structure.*

§2. Odd-dimensional invariant submanifolds of a manifold with quasi-normal (f, g, u, v, λ)-structure.

In this section we consider λ vanishes identically on the submanifold N. Then N is odd-dimensional because of Theorem 1.3.

In this case we have from (1.16)-(1.20),

(2.1) $$f_b^c f_c^a = -\delta_b^a + u_b u^a + v_b v^a,$$

(2.2) $$f_b^e f_c^d g_{ed} = g_{bc} - u_b u_c - v_b v_c,$$

(2.3) $$f_b^a u^b = 0, \quad f_b^a v^b = 0,$$

(2.4) $$u_a u^a = 1 - \sum_x \alpha_x^2, \qquad v_b v^b = 1 - \sum_x \beta_x^2,$$

(2.5) $$u_a v^a = -\sum_x \alpha_x \beta_x.$$

From (1.21) we find

(2.6) $$(\sum_x \alpha_x^2) u_b + (\sum_x \alpha_x \beta_x) v_b = 0$$

and

(2.7) $$(\sum_x \alpha_x \beta_x) u_b + (\sum_x \beta_x^2) v_b = 0,$$

from which,

(2.8) $$(\sum_x \alpha_x^2) u_b u^b + (\sum_x \alpha_x \beta_x) u_b v^b = 0,$$

or, using (2.4) and (2.5),

(2.9) $$(\sum_x \alpha_x^2)^2 + (\sum_x \alpha_x \beta_x)^2 = \sum_x \alpha_x^2.$$

Similarly, we have

$$(\sum_x \beta_x^2)^2 + (\sum_x \alpha_x \beta_x)^2 = \sum_x \beta_x^2. \tag{2.10}$$

We can easily see that $\sum_x \alpha_x^2$, $\sum_x \beta_x^2$ are globally defined functions on N [5].

We put

$$N_\alpha = \{P \in N \mid \sum_x \alpha_x^2 \neq 0\}, \quad N_\beta = \{P \in N \mid \sum_x \beta_x^2 \neq 0\}.$$

Then N_α, N_β are open in N and satisfy $N_\alpha \cup N_\beta = N$, because of the fact that N is odd-dimensional.

In N_α, we find, from (2.6)

$$u_b = -\frac{\sum_x \alpha_x \beta_x}{\sum_x \alpha_x^2} v_b, \tag{2.11}$$

from which, using (2.9),

$$u_b u^a + v_b v^a = \frac{1}{\sum_x \alpha_x^2} v_b v^a. \tag{2.12}$$

In N_β, we find, from (2.7)

$$v_b = -\frac{\sum_x \alpha_x \beta_x}{\sum_x \beta_x^2} u_b, \tag{2.13}$$

from which,

$$u_b u^a + v_b v^a = \frac{1}{\sum_x \beta_x^2} u_b u^a$$

because of (2.10).

Now we define a 1-form η_α on N in the following way: in N_α we put

$$\eta_b^{(\alpha)} = \frac{1}{\sqrt{\sum_x \alpha_x^2}} v_b \tag{2.14}$$

and in N_β

$$\eta_b^{(\beta)} = \frac{-1}{\sqrt{\sum_x \beta_x^2}} u_b. \tag{2.15}$$

In $N_\alpha \cap N_\beta$ we find, from (2.11) and (2.13)

$$(\sum_x \alpha_x \beta_x)^2 = (\sum_x \alpha_x^2)(\sum_x \beta_x^2). \tag{2.16}$$

If $\sum_x \alpha_x \beta_x = 0$ in $N_\alpha \cap N_\beta$, from (2.11) and (2.13), we have $u^a = 0$, $v^a = 0$. This shows that N is even-dimensional. So, in $N_\alpha \cap N_\beta$, $\sum_x \alpha_x \beta_x$ has no zero point. Thus we may assume that

$$\sum_x \alpha_x \beta_x > 0. \tag{2.17}$$

Therefore, in $N_\alpha \cap N_\beta$, we have

$$\eta_b^{(\alpha)} = \frac{1}{\sqrt{\sum_x \alpha_x^2}} v_b = -\frac{\sqrt{(\sum_x \alpha_x \beta_x)^2}}{\sqrt{\sum_x \alpha_x^2}\sqrt{\sum_x \beta_x^2}} u_b = -\frac{1}{\sqrt{\sum_x \beta_x^2}} u_b = \eta_b^{(\beta)}$$

because of (2.13), (2.16) and (2.17). Hence, η_b is a well defined 1-form on N.

Computing $u_b u^a + v_b v^a$, we find

$$u_b u^a + v_b v^a = \eta_b \eta^a \tag{2.18}$$

and consequently, (2.1) and (2.3) give

$$f_b{}^c f_c{}^a = -\delta_b^a + \eta_b \eta^a, \quad f_b{}^a \eta^b = 0,$$

from which,

$$-\eta^a + (\eta_b \eta^b)\eta^a = 0,$$

that is, $\eta_b \eta^b = 1$.

Thus the structure defined by $(f_b{}^a, g_{cb}, \eta_b)$ is an almost contact metric structure, that is,

$$\begin{aligned} & f_b{}^c f_c{}^a = -\delta_b^a + \eta_b \eta^a, \\ & f_c{}^e f_b{}^d g_{ed} = g_{cb} - \eta_c \eta_b, \\ & f_b{}^a \eta_a = 0, \quad f_b{}^a \eta^b = 0, \\ & \eta_a \eta^a = 1, \end{aligned} \tag{2.19}$$

where η_a is the components of the 1-form η and $\eta^b = \eta_a g^{ab}$.

Equations (2.14), (2.15) and the last equation of (2.19) say that $v_b v^b = \sum_x \alpha_x^2$ in N_α and that $u_b u^b = \sum_x \beta_x^2$ in N_β.

Now we define α and β by $\alpha^2 = \sum_x \alpha_x^2$, $\beta^2 = \sum_x \beta_x^2$, then they are globally defined functions on N and we can put

$$u^a = -\beta \eta^a, \quad v^a = \alpha \eta^a \tag{2.20}$$

because, when α or β vanishes, v^a or u^a vanishes.

From (2.18), (2.20) and $\eta_a \eta^a = 1$, we find

$$\alpha^2 + \beta^2 = 1. \tag{2.21}$$

We get from (2.20)

$$\begin{aligned} (\nabla_c u_b - \nabla_b u_c) u^a + (\nabla_c v_b - \nabla_b v_c) v^a &= \beta^2 (\nabla_c \eta_b - \nabla_b \eta_c) \eta^a + \alpha^2 (\nabla_c \eta_b - \nabla_b \eta_c) \eta^a \\ &+ \{(\nabla_c \beta)\eta_b - (\nabla_b \beta)\eta_c\} \beta \eta^a + \{(\nabla_c \alpha)\eta_b - (\nabla_b \alpha)\eta_c\} \alpha \eta^a, \end{aligned}$$

or, using (2.21),

$$(\nabla_c u_b - \nabla_b u_c) u^a + (\nabla_c v_b - \nabla_b v_c) v^a = (\nabla_c \eta_b - \nabla_b \eta_c) \eta^a. \tag{2.22}$$

If the (f, g, u, v, λ)-structure of the ambient manifold is quasi-normal, we have from (1.29), (2.18) and (2.22)

$$[f, f]_{cb}{}^a + (\nabla_c \eta_b - \nabla_b \eta_c) \eta^a - f_c{}^e f_{eb}{}^a + f_b{}^e f_{ec}{}^a = 0 \tag{2.23}$$

and

$$(\nabla_c u_b - \nabla_b u_c) \alpha_x + (\nabla_c v_b - \nabla_b v_c) \beta_x + (\eta_b h_{cex} - \eta_c h_{bex}) \eta^e = 0. \tag{2.24}$$

Transvecting (1.4) with $B_c{}^j B_b{}^i B_a{}^h$ and taking account of (2.1), we find

$$\begin{aligned} & f_c{}^e B_e{}^t (\nabla_a f_{ti}) B_b{}^i - f_b{}^e B_e{}^t (\nabla_a f_{tj}) B_c{}^j \\ & = u_c (\nabla_b u_h) B_a{}^h - u_b (\nabla_c u_h) B_a{}^h + v_c (\nabla_b v_h) B_a{}^h - v_b (\nabla_c v_h) B_a{}^h \end{aligned}$$

or, using (1.6), (1.11), (1.14) and (1.15),

$$\begin{aligned} f_c{}^e \nabla_a (f_{ti} B_e{}^t B_b{}^i) - f_b{}^e \nabla_a (f_{tj} B_e{}^t B_c{}^j) &= u_c \nabla_b u_a - u_b \nabla_c u_a + v_c \nabla_b v_a - v_b \nabla_c v_a \\ & - u_c u_h \nabla_b B_a{}^h + u_b u_h \nabla_c B_a{}^h - v_c v_h \nabla_b B_a{}^h + v_b v_h \nabla_c B_a{}^h, \end{aligned}$$

that is,

$$\begin{aligned} f_c{}^e \nabla_a f_{eb} - f_b{}^e \nabla_a f_{ec} &= u_c \nabla_b u_a - u_b \nabla_c u_a + v_c \nabla_b v_a - v_b \nabla_c v_a - u_c \textstyle\sum_x \alpha_x h_{bax} \\ & + u_b \textstyle\sum_x \alpha_x h_{cax} - v_c \textstyle\sum_x \beta_x h_{bax} + v_b \textstyle\sum_x \beta_x h_{cax}, \end{aligned}$$

or, again

$$\nabla_a(f_c{}^e f_{eb}) - 2f_{eb}\nabla_a f_c{}^e = u_c\nabla_b u_a - u_b\nabla_c u_a + v_c\nabla_b v_a - v_b\nabla_c v_a$$

by virtue of (1.21), or, using (2.19),

$$\nabla_a(-g_{cb} + \eta_c\eta_b) - 2(\nabla_a f_c{}^e) f_{eb} = u_c\nabla_b u_a - u_b\nabla_c u_a + v_c\nabla_b v_a - v_b\nabla_c v_a. \tag{2.25}$$

Substituting (2.20) into the right hand member of (2.25) and using (2.21), we obtain

$$2(\nabla_a f_c{}^e) f_{eb} = \eta_c(\nabla_a\eta_b - \nabla_b\eta_a) + \eta_b(\nabla_a\eta_c + \nabla_c\eta_a). \tag{2.26}$$

Transvecting (2.26) with η^b and using (2.19), we find

$$-\eta_c(\eta^b\nabla_b\eta_a) + \nabla_a\eta_c + \nabla_c\eta_a = 0, \tag{2.27}$$

from which, transvecting with η^a, $\eta^a\nabla_a\eta_c = 0$. Thus (2.27) becomes

$$\nabla_a\eta_c + \nabla_c\eta_a = 0, \tag{2.28}$$

that is, η^a is a Killing vector field.

Using (2.28), (2.26) can be written as

$$(\nabla_a f_c{}^e) f_{eb} = \eta_c\nabla_a\eta_b. \tag{2.29}$$

Transvecting (2.29) with $f_d{}^b$ and using (2.19), we get

$$(g_{ed} - \eta_e\eta_d)\nabla_a f_c{}^e = \eta_c f_d{}^e\nabla_a\eta_e$$

and consequently

$$\nabla_a f_{cb} = (\eta_c f_b{}^e - \eta_b f_c{}^e)\nabla_a\eta_e. \tag{2.30}$$

Thus we have

THEOREM 2.1. *Let N be an odd-dimensional invariant submanifold of a manifold with quasi-normal (f, g, u, v, λ)-structure. Then the submanifold N admits an almost contact metric structure $(f_b{}^a, g_{cb}, \eta_b)$ such that η^a is a Killing vector field and satisfies* (2.30).

§3. Odd-dimensional invariant submanifold of a manifold with (f, g, u, v, λ)-structure satisfying $\nabla_j v_i - \nabla_i v_j = 2\rho f_{ji}$.

We first prove

THEOREM 3.1. *An odd-dimensional invariant submanifold N* (dim $N>3$) *of a manifold with quasi-normal (f, g, u, v, λ)-structure satisfying*

$$\nabla_j v_i - \nabla_i v_j = 2\rho f_{ji}, \tag{3.1}$$

ρ being non-zero differentiable function, admits a Sasakian structure.

Proof. Transvecting (3.1) with $B_c{}^j B_b{}^i$, we find

$$\nabla_c v_b - \nabla_b v_c = 2\rho f_{cb}. \tag{3.2}$$

Substituting (3.2) into (2.24), we obtain

$$(\nabla_c u_b - \nabla_b u_c)\alpha_x + 2\rho\beta_x f_{cb} + (\eta_b h_{cex} - \eta_c h_{bex})\eta^e = 0, \tag{3.3}$$

from which, transvecting η^b,

$$\eta^b(\nabla_c u_b - \nabla_b u_c)\alpha_x + h_{cex}\eta^e - (h_{bex}\eta^b\eta^e)\eta_c = 0. \tag{3.4}$$

Suppose that there exists a point P at which $\alpha(P) = 0$, then $\alpha_x(P) = 0$ for all x. Consequently we have at P

$$h_{cex}\eta^e=(h_{bex}\eta^b\eta^e)\eta_c.$$

Thus (3.3) implies that $\beta_x(P)=0$ and this, together with (1.22), shows that the submanifold N is even-dimensional. Therefore, α is non-zero.

If we substitute $v_b=\alpha\eta_b$ into (3.2), we get

$$2\alpha(\nabla_c\eta_b)+(\nabla_c\alpha)\eta_b-(\nabla_b\alpha)\eta_c=2\rho f_{cb} \tag{3.5}$$

by virtue of (2.28), from which, transvecting η^b,

$$\nabla_c\alpha=A\eta_c, \tag{3.6}$$

where we have put $A=\eta^a\nabla_a\alpha$. Thus (3.5) becomes

$$\alpha\nabla_c\eta_b=\rho f_{cb}. \tag{3.7}$$

Differentiating (3.6) covariantly, we find

$$\nabla_b\nabla_c\alpha=A\nabla_b\eta_c+(\nabla_bA)\eta_c,$$

from which,

$$A(\nabla_b\eta_c-\nabla_c\eta_b)+(\nabla_bA)\eta_c-(\nabla_cA)\eta_b=0. \tag{3.8}$$

Transvecting (3.8) with f^{bc} and using (3.7), we have $(n-1)A=0$ which, together with (3.6), implies that $\nabla_c\alpha=0$. Thus α is non-zero constant.

Substituting (3.7) into (2.30) and using (2.19), we find

$$\alpha\nabla_a f_{cb}=\rho(\eta_c g_{ba}-\eta_b g_{ca}), \tag{3.9}$$

from which, $\alpha f_{acb}=0$. Since α is non-zero constant, we have $f_{acb}=0$.

We have from (3.7)

$$\alpha(\nabla_c\eta_b-\nabla_b\eta_c)=2\rho f_{cb}.$$

Differentiating the last equation covariantly, we find

$$\alpha(\nabla_a\nabla_c\eta_b-\nabla_a\nabla_b\eta_c)=2\rho\nabla_a f_{cb}+2f_{cb}\nabla_a\rho,$$

from which, using Ricci identity and $f_{acb}=0$,

$$f_{cb}\nabla_a\rho+f_{ba}\nabla_c\rho+f_{ac}\nabla_b\rho=0. \tag{3.10}$$

Transvecting (3.10) with η^b, we get

$$(n-1)\eta^b\nabla_b\rho=0. \tag{3.11}$$

Transvecting (3.10) with f^{cb} again and taking account of (3.11), we find $(n-3)\nabla_a\rho=0$ and consequently ρ=constant. Thus submanifold admits a Sasakian structure. This completes the proof of the theorem.

If the (f, g, u, v, λ)-structure of the ambient manifold is normal and satisfies $\nabla_j v_i-\nabla_i v_j=2\rho f_{ji}$, then we can similarly derive (3.7) and (3.9). Thus we have

THEOREM 3.2. *An odd-dimensional invariant submanifold N* (dim $N>3$) *of a manifold with normal (f, g, u, v, λ)-structure satisfying*

$$\nabla_j v_i-\nabla_i v_j=2\rho f_{ji},$$

ρ being non-zero differentiable function, admits a Sasakian structure.

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