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MEAN ERGODIC THEOREM TO HOLD FOR A HARMONIZABLE PROCESS

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1. Preliminaries

K. Nagabhushanam [4] derived a mean ergodic theorem to hold for a harmonizable process of discrete parameter, using the covariance spectrum of the process.

T. Kawata [3] derived a mean ergodic theorem to hold for general nonstationary processes of continuous parameter, using the two dimensional Wiener formula and the 2-transform [2]. This paper deals with an application of the theorem in [3] to get a theorem on a harmonizable process of continuous parameter which is an analogue of the theorem in [4].

Kawata's theorem in [3] is as follows:

Let us suppose that a stochastic process

$$X(x,\omega),\quad -\infty<x<\infty,\quad \omega\in\Omega$$

satisfies the following conditions,

(1) $X(x,\omega),\ -\infty<x<\infty,\ \omega\in\Omega$ *is measurable and separable,*

(2) $E|X(x,\omega)|^2<\infty$ *for every* x,

(3) $\int_a^b E|X(x,\omega)|^2<\infty$ *for every finite interval* (a, b),

(4) $EX(x,\omega)=0$ *for every* x, *and*

(5) *The covariance function*

$$\rho(s,t)=EX(s,\omega)\overline{X(t,\omega)}$$

is continuous in $-\infty<s,\ t<\infty$;

$$\frac{1}{ST}\int_0^T\int_0^S\rho(x,y)\,dx\,dy \tag{1.1}$$

converges as $S\to\infty,\ T\to\infty$, *and* $S\to-\infty,\ T\to-\infty$; *and*

$$\left|\int_0^T\int_0^S|\rho(x,y)|\,dx\,dy\right|\le D|ST| \tag{1.2}$$

for every pair of real numbers S *and* T.

Then

$$\underset{T\to\infty}{\mathrm{l.i.m.}}\frac{1}{2T}\int_{-T}^{T}X(x,\omega)\,dx=\underset{\varepsilon\to0}{\mathrm{l.i.m.}}\frac{1}{\varepsilon\sqrt{2\pi}}\,\Delta_\varepsilon^2Y_2(-\varepsilon,\omega), \tag{1.3}$$

where

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$$\Delta_\varepsilon^2 Y_2(-\varepsilon,\omega)=Y_2(\varepsilon,\omega)-2Y_2(0,\omega)+Y_2(-\varepsilon,\omega),$$

and $Y_2(y,\omega)$ *is the 2-transform of* $X(x,\omega)$ *(for the definition, see [3]) which exists, and* (1.3) *means that both sides exist and are equal to each other.*

2. Main theorem

For a harmonizable process

$$X(x,\omega),\quad -\infty<x<\infty,\quad \omega\in\Omega, \tag{2.1}$$

with the covariance function

$$\rho(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{ix\lambda-iy\lambda'}d^2F(\lambda,\lambda') \tag{2.2}$$

where $F(\lambda,\lambda')$ is the spectral function, the conditions (2), (3) and (5) obviously hold. Therefore our theorem is the following:

THEOREM. *For a harmonizable process* (2.1) *satisfying conditions* (1) *and* (4), *the necessary and sufficient condition for the mean square convergence of* $\frac{1}{2T}\int_{-T}^{T}X(x,\omega)dx$ *to zero is that the spectral function is continuous at the origin.*

Proof. Firstly we will prove that the convergence of (1.1) is satisfied for our harmonizable process.

$$\frac{1}{ST}\int_0^T\int_0^S\rho(x,y)dxdy=\frac{1}{ST}\int_0^T\int_0^S\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i\lambda x-i\lambda' y}d^2F(\lambda,\lambda')dxdy$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i\lambda S}-1}{i\lambda S}\cdot\frac{e^{-i\lambda' T}-1}{-i\lambda' T}d^2F(\lambda,\lambda').$$

As $F(\lambda,\lambda')$ is continuous at the origin we can fix a positive number η for any small $\delta>0$, such that in the square Q_δ with $|\lambda|<\eta$, $|\lambda'|<\eta$, the following inequality holds,

$$\int\int_{(\lambda,\lambda')\in Q_\delta}|d^2F(\lambda,\lambda')|<\delta.$$

In the complements $\tilde{Q}_\delta$ of Q_δ in the (λ,λ')-plane, at least one of $|\lambda|$ and $|\lambda'|$ is not less than $\eta>0$, so that we can say

that
$$\frac{e^{i\lambda S}-1}{i\lambda S}\cdot\frac{e^{-i\lambda' T}-1}{-i\lambda' T}\to 0$$

as $S\to\infty$, $T\to\infty$, and $S\to-\infty$, $T\to-\infty$.

As $\left|\frac{e^{i\lambda S}-1}{i\lambda S}\cdot\frac{e^{-i\lambda' T}-1}{-i\lambda' T}\right|$ is uniformly bounded for all λ,λ', S, and T, we have

$$\left|\frac{1}{ST}\int_0^T\int_0^S\rho(x,y)dxdy\right|$$
$$\leqq\left|\int\int_{(\lambda,\lambda')\in Q_\delta}\frac{e^{i\lambda S}-1}{i\lambda S}\cdot\frac{e^{-i\lambda' T}-1}{-i\lambda' T}d^2F\right|+\left|\int\int_{(\lambda,\lambda')\in\tilde{Q}_\delta}\right|$$

in which, on the right hand side, the first term is smaller than $\delta>0$, and the second becomes arbitrarilly small when the absolute values of S and T are sufficiently large. Thus we have proved that

$$\frac{1}{ST}\int_0^T\int_0^S \rho(x,y)dxdy$$

converges to zero as $S\to\infty$, $T\to\infty$, and $S\to-\infty$, $T\to-\infty$.

In the above proof we used the continuity of F at the origin. But even without this condition we can prove the convergence of (1.1), in which case (1.1) converges to $F(+0,+0)-F(+0,-0)-F(-0,+0)+F(-0,-0)$.

Secondly we will prove that condition (1.2) holds.

$$\left|\int_0^T\int_0^S |\rho(x,y)|dxdy\right| \leq \left|\int_0^T\int_0^S\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |e^{i\lambda x-i\lambda' y}||d^2F|\right|$$

$$\leq \left|\int_0^T\int_0^S\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |d^2F|\right|$$

$$\leq \left|\int_0^T\int_0^S D dxdy\right| = D|ST|,$$

where

$$\int\int |d^2F| \leq D.$$

Thus we have proved that our process satisfies conditions (1.1) and (1.2). Therefore we can conclude that (1.3) holds. It remains lastly to prove that the continuity of $F(\lambda,\lambda')$ at the origin is equivalent to

$$\underset{\varepsilon\to0}{\text{l.i.m.}}\ \frac{1}{\varepsilon}\Delta_\varepsilon^2 Y(-\varepsilon,\omega)=0.$$

From [3] we have

$$E\left|\frac{1}{\varepsilon}\Delta_\varepsilon^2 Y_2(-\varepsilon,\omega)\right|^2=\frac{1}{2\pi\varepsilon^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho(x,y)\frac{\sin^2(\varepsilon x/2)}{(x/2)^2}\ \frac{\sin^2(\varepsilon y/2)}{(y/2)^2}dxdy$$

$$=\frac{1}{2\pi\varepsilon^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i\lambda x}\sin^2(\varepsilon x/2)}{(x/2)^2}\cdot\frac{e^{-i\lambda' y}\sin^2(\varepsilon y/2)}{(y/2)^2}dxdy\ d^2F(\lambda,\lambda')$$

$$=\frac{2}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d^2F\int_{-\infty}^{\infty}e^{i2\lambda x/\varepsilon}\frac{\sin^2 x}{x^2}dx\int_{-\infty}^{\infty}e^{-i2\lambda' y/\varepsilon}\frac{\sin^2 y}{y^2}dy$$

$$=\frac{2}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}a(\lambda)\overline{a(\lambda')}\ d^2F,$$

where

$$a(\lambda)=\int_{-\infty}^{\infty}e^{i2\lambda x/\varepsilon}\frac{\sin^2 x}{x^2}dx.$$

Write

$$|a(\lambda)|\leq\left|\int_{|x|<\delta}\right|+\left|\int_{|x|\geq\delta}\right|$$

for any fixed small $\delta>0$, where on the right hand side the first term is smaller than 2δ and the second term approaches zero as $\varepsilon\to 0$ uniformly for $|\lambda|\geq\eta$, η being any fixed positive constant because we can apply the Riemann-Lebesgue theorem. Thus $|a(\lambda)|$ approaches zero as $\varepsilon\to 0$, for $|\lambda|\geq\eta$.

Now let us decompose the (λ, λ')-plane into two parts:

$$Q_\delta=\{(\lambda, \lambda');\ |\lambda|<\eta,\ |\lambda'|<\eta\} \text{ and}$$

$\tilde{Q}_\delta$=complement of Q_δ in(λ, λ')-plane.

Then we have

$$E\left|\frac{1}{\varepsilon}\Delta_\varepsilon^2 Y_2(-\varepsilon, \omega)\right|^2\leq\frac{2}{\pi}\left|\int\int_{(\lambda,\lambda')\in\tilde{Q}_\delta} a(\lambda)\overline{a(\lambda')}\, d^2F(\lambda, \lambda')\right|$$
$$+\frac{2}{\pi}\left|\int\int_{(\lambda,\lambda')\in Q_\delta} a(\lambda)\overline{a(\lambda')}\, d^2F(\lambda, \lambda')\right|$$

On the right hand side the second term appoaches zero as $\varepsilon\to 0$, and the first term becomes arbitrarily small since $|a(\lambda)|\leq 2\pi$, if and only if $F(\lambda, \lambda')$ is continuous at the origin. q. e. d.

3. Remarks

"The continuity of the spectral functions $F(\lambda, \lambda')$ at the origin" in our theorem corresponds to "the continuity of the spectrum $W(\lambda)$ at the origin" in the theorem of K. Nagabhushanam [4], where $W(\lambda)$ is equal to

$$F(\pi, \pi)-F(-\lambda, -\lambda)+W_1(\lambda),$$

and $W_1(\lambda)$ is a function equal to zero except at the points $(-\pi, \pi)$ and $(\pi, -\pi)$. Therefore the continuity of $W(\lambda)$ at the origin is not other than the continuity of $F(\lambda, \lambda)$ at the origin.

We see also that our theorem is an analogue of the well known Slutky's theorem [1].

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