J. Korean Math. Soc.
Vol. 9, No. 1, 1972

SEQUENTIALLY COMPLETE SPACES

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1. Introduction.

The definition of a uniform structure we shall adopt is the one defined by means of pseudometrics. All topological spaces are assumed to be uniformizable and Hausdorff, and we use the word "space" always to mean a completely regular T_1-space.

In a complete uniform space, Cauchy sequences are exactly those sequences which are convergent. Accordingly, a sufficient condition for a space X to admit a sequentially complete uniform structure, i. e., a uniform structure under which Cauchy sequences converge, is that X can be embedded in a topologically complete space Y in such a way that any sequence of points in X converges to a point of X if it ever converges in Y. Moreover, since X must have a completion with respect to its largest admissible structure, this condition is necessary as well. Our goal in this article is to present an elemantary and self-contained proof that the Stone-Cech compactification βX can play the role of such a complete envelope in order to discriminate topologically sequentially complete spaces. To be precise, we have

THEOREM 1. *The following are equivalent for any space* X:

(1) X *admits a sequentially complete uniform structure,*

(2) X *admits a sequentially complete precompact uniform structure,*

(3) *No sequence of points in* X *converges to a point of* $\beta X-X$.

2. Some equivalent conditions.

Throughout, U^* will denote the (unique) uniform structure of βX relativized to X.

LEMMA 1. *The statement* (2) *is equivalent to*

(2′) X *is sequentially complete with respect to the structure* U^*.

Proof. U^* is the largest precompact structure admissible to X.

As usual, let $C(X)$ denote the ring of real functions continuous on X, and let $C^*(X)$ denote the subring of $C(X)$ consisting of bounded functions. It should be noted that the set of pseudometrics of the form $d_f(x,y)=|f(x)-f(y)|$, $f\in C^*(X)$, constitutes a subbase for the structure U^*. If the functions f are taken from the whole ring $C(X)$, then the corresponding pseudometrics of course generate a uniform structure admissible for X. By completing X relative to this structure, we obtain the Hewitt realcompactification υX, which is the largest subspace of βX subject to the condition: every $f\in C(X)$ extend to some $g\in C(\upsilon X)$.

Received by the editors Mar. 10, 1972.

LEMMA 2. *The statement* (3) *is equivalent to*
(3′) *No sequence of points in* X *converges to a point of* $\upsilon X-X$.

Proof. To prove the nontrivial part, let $\{x_n\}$ be a sequence of points in X which converges to a point p in $\beta X-X$. In order to assert that p is in υX, it suffices to show that $f^*(p)\neq\infty$ for each f in $C(X)$, where f^* denotes the Stone extension of f. Suppose contrary that we had $f^*(p)=\infty$ for some f in $C(X)$. By taking $-f$ in place of f if necessary, we can take a subsequence $\{x_{n(k)}\}$ of the given sequence such that $\{f(x_{n(k)})\}$ is an unbounded strictly increasing sequence of real numbers. Pick a homeomorphism h of the real line onto itself that sends $f(x_{n(k)})$ to $k\pi$, $k=1, 2, \cdots$, and define $g\in C^*(X)$ by $g(x)=\cos(hf(x))$. Then we have $g(x_{n(k)})=(-1)^{k-1}$, and the sequence $\{g(x_n)\}$ fails to converge. Since the sequence $\{x_n\}$ converges to p, this leads to the contradiction that g cannot be extended continuously to a function defined on βX.

3. Precompact structures.

The following seems to be generally accepted. Its validity rests on the fact that if d is a continuous pseudometric on X then $1\wedge d(p, x)$ belongs to $C^*(X)$ for each fixed p in X.

LEMMA 3. *If* D *is an admissible uniform structure for a space* X, *so is* $D\cap U^*$.

The most crucial part of the main Theorem 1 is furnished by the following

LEMMA 4. *Let* D *be a uniform structure admissible to the space* X. *If* D *is sequentially complete, so is* $D\cap U^*$.

Proof. Let $\{x_n\}$ be a sequence of points in X that fails to be a Cauchy sequence with respect to the structrue D. There exist a pseudometric $d\in D$ and a positive number ε such that for any integer k there are integers m and n with $m, n>k$ and $d(x_m, x_n)>\varepsilon$. Accordingly, the sequence $\{x_n\}$ has a subsequence $\{x_{n(p)}\}$ with the property that $d(x_{n(2p-1)}, x_{n(2p)})>\varepsilon$. For the sake of previty, we assume with no loss of generality that $\{x_n\}$ has this property, i.e., $d(x_{2n-1}, x_{2n})>\varepsilon$ for all n. We may also suppose that d is a bounded pseudometric.

For each positive integer p, let N_p be the set of positive integers n with $d(x_{2p-1}, x_{2n-1})<\varepsilon/3$. We divide the case according as (1) N_p is an infinite set for some p, or (2) each N_p is a finite set.

Case 1. Let p be an integer with N_p infinite, let A denote the set of points x_{2n-1} with $n\in N_p$, and let d^* be the pseudometric defined by

$$d^*(x, y)=|d(A, x)-d(A, y)|$$

for x, y in X. This is indeed a well defined pseudometric in U^* as the map $x\to d(A, x)$ belongs to $C^*(X)$. Moreover, since the relation

$$|d(a, x)-d(a, y)|\leq d(x, y)$$

is valid for all a, x and y in X, it follows that $d^*(x,y)\leq d(x,y)$, and d^* belongs to D as well. Since $m, n\in N_p$ implies

$$\begin{aligned} d(x_{2m-1}, x_{2n}) &\geq |d(x_{2n-1}, x_{2n})-d(x_{2m-1}, x_{2n-1})| \\ &\geq |d(x_{2n-1}, x_{2n})-(d(x_{2m-1}, x_{2p-1})+d(x_{2p-1}, x_{2n-1}))| \\ &> \varepsilon/3, \end{aligned}$$

we have

$$d^*(x_{2n-1}, x_{2n})=|d(A, x_{2n-1})-d(A, x_{2n})|=d(A, x_{2n})\geq\varepsilon/3$$

for each n in N_p. This shows that $\{x_n\}$ fails to be a Cauchy sequence with respect to the structure $D\cap U^*$ because N_p must be cofinal in the set of natural numbers.

Case 2. Each N_p is a finite set in this case, and we can construct by induction an infinte set L of natural numbers subject to the condition that p does not belong to N_q for any pair of distinct members p and q of L. Let L be expressed as the disjoint sum of infinite subsets J and K, let A denote the set of points x_{2j-1} with j in J, and let d^* be defined by

$$d^*(x,y)=|d(A,x)-d(A,y)|$$

for x, y in X. As in Case 1, d^* is a well defined pseudometric in $D\cap U^*$. Moreover, since $d(x_{2p-1}, x_{2q-1})\geq\varepsilon/3$ for any distinct p, q in L by the very definition of L, we have

$$d^*(x_{2j-1}, x_{2k-1})=d(A, x_{2k-1})\geq\varepsilon/3$$

for any j in J and k in K. Because J and K are both infinite, it follows that $\{x_n\}$ fails to be a Cauchy sequence relative to the structure $D\cap U^*$. This completes the proof of Lemma 4 as it is now clear that a sequence of points in X is a Cauchy sequence relative to D if and only if it is relative to $D\cap U^*$.

4. Proof of Theorem 1.

We now proceed to show that the conditions (1), (2) and (3) of Theorem 1 are equivalent. Since (2) follows from (1) by Lemmas 3 and 4, all we have to do is to prove that (2) implies (3) and (3) implies (1).

(2) implies (3): If a sequence of points in X converges to a point of βX, it is a Cauchy sequence relative to the structure U^*. Accordingly, it converges to a point of X as X must be sequentially complete with respect to the structure U^* by Lemma 1. Since βX is a Hausdorff space, this means that no sequence of points in X can converge to a point of βX that fails to be in X.

(3) implies (1): If a sequence of points in X fails to converge in X, it also fails to converge in βX by (3). This, however, means that the sequence fails to be a Cauchy sequence relative to U^* because the structure U^* of X is inherited from the compact, and hence complete, uniform space βX. We have proved that U^* is a sequentially complete uniform structure, which clearly is admissible to X.

As a straightforward consequence of Theorem 1, we have

COROLLARY. *If a noncompact space admits a sequentially complete uniform structure, it admits a sequentially complete uniform structure that fails to be a complete structure.*

This corollary, combined with the results of the next section, provides us with ample supply of sequentially complete uniform spaces that are not complete.

5. Conditions under which spaces admit sequentially complete structures.

Thus far, we have been concerned with characterizing spaces which admit sequentially complete uniform structures. Our purpose in this final section is to seek some topological properties of spaces ensuring sequential completeness.

The first of our results in this section is a rather trivial one. However, its proof given here does not depend upon topological completeness of realcompact spaces since Lemma 2 is proved without using any uniformity concept; the only property of υX we have used in proving Lemma 2 is that υX is the largest subspace of βX over which every $f \in C(X)$ has a real valued continuous extension.

THEOREM 2. *Every realcompact space admits a sequentially complete uniform structure.*

Proof. Immediate from Theorem 1 and Lemma 2 because $X = \upsilon X$ for X realcompact.

Let Y be a noncompact but locally compact space embedded in a space X. If a function f in $C(Y)$ fulfills the condition

(*) for any positive number ε, there is a compact subset K of Y such that $|f(x)-f(y)|<\varepsilon$ for any x and y in $Y-K$,

one easily checks that f extends to a function in $C(X)$. If, conversely, every member of $C(X)$ satisfies the condition (*) when restricted to Y, we say that Y is *inadequately embedded in* X. In other words, Y is inadequately embedded in X if and only if members of $C(X)$ restricted to Y form a ring isomorphic with $C(Y^*)$ where Y^* denotes the one point compactification of Y. In this terminology, the equivalence between (1) and (3) in Theorem 1 is converted into the following form:

THEOREM 3. *A space X admits a sequentially complete uniform structure if and only if X does not have a countably infinite discrete closed subspace inadequately embedded in X.*

One may then invoke the Tietze extension theorem to obtain the following known result.

THEOREM 4. *Every normal space admits a sequentially complete uniform structure.*

REMARK. Theorem 3 does not require that a topologically sequentially complete space X be normal, nor even that bounded continuous functions on a countably infinite discrete closed subset be extendable over X. To see this, let the space N of natural numbers be expressed as the disjoint sum of two copies N_1 and N_2 of itself. Then, βN

is the disjoint sum of βN_1 and βN_2 both homeomorphic with βN. Let Y be the space obtained from βN by identifying a point x_1 in $\beta N_1 - N_1$ to a point x_2 in $\beta N_2 - N_2$, let W^* denote the space of ordinals not exceeding the first uncountable ordinal Ω, and let $W = W^* - \{\Omega\}$. Then, the subspace

$$X = (Y \times W) \cup (N \times \{\Omega\})$$

of $Y \times W^*$ has $Y \times W^*$ as its Stone-Cech compactification. Now, if D is an infinite discrete closed subset of X, all but finitely many points of D lie in $N \times \{\Omega\}$ by countable compactness of $Y \times W$. Hence, we may suppose that D intersects, say, N_1 on an infinite set E. Since every function in $C^*(E)$ extends over X, it follows that D is not inadequately embedded in X, and X admits a sequentially complete uniform structure by Theorem 3. However, the function in $C^*(N \times \{\Omega\})$ taking values i on $N_i \times \{\Omega\}$, $i = 1, 2$, fails to extend continuously over X.

The following analogue of Theorem 4 is also a consequence of Theorem 3.

THEOREM 5. *Every countably paracompact space admits a sequentially complete uniform structure.*

Proof. Let X be a countably paracompact space and let D be a countable discrete subspace of X which is closed in X. By Theorem 3, it suffices to show that every f in $C(D)$ extends over X. To see this, let $\{U_d\}$ be a locally finite collection of open subsets of X indexed by the points of D such that the only point of D contained in U_d is d, and let f_d be a function in $C(X)$ such that $f_d(d) = f(d)$ but f_d vanishes outside U_d for each d in D. Local finiteness of $\{U_d\}$ implies that the pointwise sum of the functions f_d is a well defined function belonging to $C(X)$. This completes the proof of Theorem 5.

REMARK. Countable metacompactness cannot replace countable paracompactness in Theorem 5. In fact, under the notation of the remark that follows Theorems 3 and 4, the Tychonoff plank $N^* \times W^* - (p, \Omega)$ gives a counter example to the tempting conjecture, that countable metacompactness imply sequential completeness, where N^* denotes the one point compactification of N with point at infinity designated by p.

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