J. Korean Math. Soc.
Vol. 9, No. 1, 1972

SOME PROPERTIES IN THE PSEUDO-RIEMANNIAN SPACE WITH THE SCHOUTENEAN CONNECTION

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PRELIMINARY: The auther has studied on some properties in the semi-symmetric metric space with Schoutenean connection [1]. In the present paper he continuously obtains some propositions on the geodesics and hypersurface embedded in it.

[1] The followings had been seen already, that the pseudo-Riemannian space R_n^* may be induced to the space R_n under the condition whose metric is of the form

$$ds^2 = a_{ij}dx^i dx^j$$

and admitting the fundamental semi-symmetric torsion of Schouten:

$$(1)\qquad T_{ij}{}^h = \delta_i{}^h \frac{\partial U}{\partial x^j} - \delta_j{}^h \frac{\partial U}{\partial x^i}, \qquad T_{ijk} = a_{ik}\frac{\partial U}{\partial x^j} - a_{jk}\frac{\partial U}{\partial x^i},$$

where U be the component of the torsion vector in R_n^*.

Here we put the metric as follows:

$$(2)\qquad ds^2 = e^{2U} g_{ij} dx^i dx^j, \qquad a_{ij} = e^{2U} g_{ij},$$

then in this case the affine connection should be given in the next formula:

$$(3)\qquad \Gamma_{ij}{}^h = \left\{ {}_i{}^h{}_j \right\} + \delta_i{}^h \frac{\partial U}{\partial x^j}$$

where $\left\{ {}_i{}^h{}_j \right\}$ is the Christoffel's symbol with respect to g_{ij}.

Further we could already obtained [1]

$$(4)\qquad R^h{}_{ijl} = G^h{}_{ijl}$$

where $G^h{}_{ijl}$ be the curvature tensor of the Riemannian space R_n with the metric

$$(5)\qquad d\sigma^2 = g_{ij} dx^i dx^j.$$

Then we can see that the pseudo-Riemannian space R_n^* is conformally equivalent with the Riemannian space R_n admitting the same curvature tensor. And in this case the following can easily be seen:

$$(6)\qquad R_{ihjl} = e^{2U} G_{ihjl}.$$

Received by the editors Mar. 10, 1972.

From (4) and (6) various results have been obtained [5], whereas if G_{ihjl} vanishes, the so-called absolute parallelism is to be induced in the pseudo-Riemannian space.

Now we examine the existence condition

$$\text{(7)} \qquad \frac{d^2x^i}{ds^2}+\left\{ {i \atop k\,h} \right\}\frac{dx^k}{ds}\frac{dx^h}{ds}+\frac{dx^i}{ds}\frac{\partial U}{\partial x^k}\frac{dx^k}{ds}=0$$

of the straight line in R_n^* of which metric is given by (2).

On the other hand, the conditions for the geodesics is given as follows [4],

$$\text{(8)} \qquad \frac{d^2x^i}{ds^2}+\left\{ {i \atop k\,h} \right\}\frac{dx^k}{ds}\frac{dx^h}{ds}+2\frac{dx^i}{ds}\frac{\partial U}{\partial x^k}\frac{dx^k}{ds}-g^{ik}e^{-2U}\frac{\partial U}{\partial x^k}=0.$$

Here, through a few steps of calculation we hold, in our space the following:

If the geodesics is auto-parallel, then it is in fact the straight line.

Paying attention to (5), let us consider the following:

$$\text{(9)} \qquad \frac{d^2x^i}{d\sigma^2}+\left\{ {i \atop k\,h} \right\}\frac{dx^k}{d\sigma}\frac{dx^h}{d\sigma}=\kappa n^i$$

where κ, n^i be the principal curvature and unitary vector of the principal normal respectively.

Applying the results related with the godesics in the semi-symmetric metric connection space to the following curve:

$$\text{(10)} \qquad \frac{d^2x^i}{d\sigma^2}+\left\{ {i \atop k\,h} \right\}\frac{dx^k}{d\sigma}\frac{dx^h}{d\sigma}-\left(g^{ik}-\frac{\partial x^i}{\partial\sigma}\frac{\partial x^k}{\partial\sigma}\right)\frac{\partial U}{\partial x^k}=0,$$

we obtain the next formulae:

$$\text{(11)} \qquad \begin{cases} \kappa n^i=\left(g^{ik}-\dfrac{dx^i}{ds}\dfrac{dx^k}{ds}\right)\dfrac{\partial U}{\partial x^k}, \\ \kappa^2=\left(g^{kh}-\dfrac{dx^k}{ds}\dfrac{dx^h}{ds}\right)\dfrac{\partial U}{\partial x^k}\dfrac{\partial U}{\partial x^h}, \\ \kappa=\dfrac{\partial U}{\partial x^k}n^k. \end{cases}$$

Covariant differentiating the 1-st and 2-nd formulae with respect to σ, we have

$$\text{(12)} \qquad \frac{d\kappa}{d\sigma}=n^k\frac{dx^h}{d\sigma}\left[\frac{\partial^2 U}{\partial x^k\partial x^h}-\frac{\partial U}{\partial x^k}\frac{\partial U}{\partial x^h}-\left\{ {l \atop k\,h} \right\}\frac{\partial U}{\partial x^l}\right],$$

$$\text{(13)} \qquad \kappa\left(\frac{\delta n^i}{d\sigma}+\kappa\frac{dx^i}{d\sigma}\right)=\left(g^{ik}-\frac{dx^i}{d\sigma}\frac{dx^k}{d\sigma}-n^in^k\right)\frac{dx^h}{d\sigma}\cdot\left[\frac{\partial^2 U}{\partial x^k\partial x^h}-\left\{ {l \atop k\,h} \right\}\frac{\partial U}{\partial x^l}\right],$$

where $\dfrac{\delta n^i}{d\sigma}$ indicates the covariant derivatives of the principal normal n^i with respect to σ.

Considering the interior of the bracket of the right hand side of (12), we obtain the next proposition.

PROPOSITION 1. *If there exists the fundamental tensor field*:

$$\begin{cases} g_{kh}\xi=\dfrac{\partial^2 U}{\partial x^k \partial x^h}=\{{}_k{}^l{}_h\}\dfrac{\partial U}{\partial x^l}-\dfrac{\partial U}{\partial x^k}\dfrac{\partial U}{\partial x^h}, \\ d\sigma^2=g_{kh}dx^k dx^h, \end{cases} \tag{14}$$

then the principal curvature κ *is constant.*

[2] We consider the equations:

$$x^i=f^i(u^1, u^2, \cdots, u^{n-1}), \qquad i=1, 2, \cdots, n.$$

of the hypersurface in R_n^*, that is admitted the semi-symmetric Schoutenean connection. The linear metric element on this hypersurface is given as follows:

$$ds^2=g_{ij}\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta}du^\alpha du^\beta,$$
$$\alpha,\ \beta=1, 2, \cdots, n-1, \qquad i,\ j=1, 2, \cdots, n.$$

For the differential equations of the geodesics, we have

$$g_{ij}\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta}\frac{d^2u^\beta}{ds^2}+g_{ij}\{{}_k{}^j{}_h\}\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^k}{\partial u^\beta}\frac{\partial x^h}{\partial u^\gamma}\frac{du^\beta}{ds}\frac{du^\gamma}{ds}$$
$$+g_{ij}\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}\frac{\partial x^j}{\partial u^\gamma}\frac{du^\beta}{ds}\frac{du^\gamma}{ds}=0, \tag{15}$$

and along this curve:

$$\frac{dx^i}{ds}=\frac{\partial x^i}{\partial u^\alpha}\frac{du^\alpha}{ds},$$

$$\frac{d^2x^i}{ds^2}=\frac{\partial x^i}{\partial u^\alpha}\frac{d^2u^\alpha}{ds^2}+\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds},$$

then, we can rewrite (15) as follows

$$\frac{d^2x^i}{ds^2}+\{{}_k{}^i{}_h\}\frac{dx^k}{ds}\frac{dx^h}{ds}+V^i-\frac{dx^i}{ds}V_k\frac{dx^k}{ds}=\kappa\nu^i, \tag{16}$$

where $V_i=\dfrac{\partial U}{\partial x^i}$ and κ, ν^i be the curvature and unitary vector of the principal direction respectively.

Contracting (16) with $g_{ij}\dfrac{\partial x^j}{\partial u^\gamma}$ and putting the unitary vector of the tangent to be t_i, we have

$$V_i\frac{\partial x^i}{\partial u^\gamma}-t_i V_k\frac{dx^k}{ds}\frac{\partial x^i}{\partial u^\gamma}=\kappa\nu_i\frac{\partial x^i}{\partial u^\gamma},$$

i. e.,

(17) $$\left(V_i - t_i V_k \frac{dx^k}{ds} - \kappa\nu_i\right)\frac{\partial x^i}{\partial u^\gamma} = 0.$$

This means that the interior of the bracket be orthogonal to $\frac{\partial x^i}{\partial u^\gamma}$ on the hypersurface, so that its direction is the same with the normal of the hypersurface.

Setting the unitary vector of the normal to be N_i, then we have

$$V_i - t_i V_k \frac{dx^k}{ds} - \kappa\nu_i = \lambda N_i,$$

i. e.,

$$V_i = V_k \frac{dx^k}{ds} t_i + \kappa\nu_i + \lambda N_i.$$

Thus we have derived the following:

PROPOSITION 2. *Concerning of a geodesics, torsion vector of our space on a point of the hypersurface can be expressed by means of the* 3-*frame* (t_i, ν_i, N_i) *and the curvature of the curve and the projection curve of the torsion vector onto the tangent plane could be gotten through the parallelopiped constructed with this* 3-*frame.*

References

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