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A CHARACTERIZATION OF RIGHT ORDERS IN q-RINGS

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1. Introduction.

Throughout this paper, we assume that every ring has an identity $1\neq 0$, and every module is unitary unless mentioned otherwise. An overring Q of R is called a (right) *classical quotient ring* of R if and only if every *regular* element (=non-zero divisor) of R has a two-sided inverse in Q and every element of Q has the form ab^{-1}, where $a\in R$, $b(\neq 0)$ regular in R. In this case the subring R is called a *right order* in Q. It may be remarked that Q is a (right) *quotient ring* of R in the sense of Johnson ([4], p. 894) if the subring R is a right order in Q. And that the right R-module Q_R is an essential extension of R_R.

A ring R of which every right ideal is quasi-injective is called q-ring ([2], p. 73). In this paper it is shown that a necessary and sufficient condition of a given ring R to be embedded as a right order in a q-ring Q is that R satisfies the following conditions:

(1) $T(E(R)/R)=E(R)/R$,

(2) for any regular b in R, $bE(R)\supseteq R$,

(3) every large right ideal of R which has the form $L\cap R$ where L is a large right ideal of $E(R)$ as ring structure is two-sided,

(4) for every large right ideal L of $E(R)$ and every regular element b in R, $L\subseteq bL$.

2. q-rings and injective hulls.

If a ring Q is a right classical quotient ring of a ring R, the left R-module ${}_RQ$ is flat, i.e., functor $\otimes_R Q$ is exact. To prove the flatness by ([5], Proposition 1, p. 132), it is sufficient to show that if I be a right ideal of R, then $I\otimes_R Q\cong IQ$ canonically. For if $aq=0$, $q=cd^{-1}$, i.e., $aq=a(cd^{-1})=(ac)d^{-1}=0$, then $ac=0$ and $a\otimes q=a\otimes cd^{-1}=ac\otimes d^{-1}=0$. The flatness of ${}_RQ$ shall be used later.

If M_R be an R-module, let

$$T(M)=\{m\in M \mid \text{there exists regular } b \text{ in } R,\ mb=0\}$$

J. P. Jans ([3], Lemma 1, p. 37) characterizes $T(M)$ using the concept of quotient ring of R as follows.

PROPOSITION 1. *If R has a right classical quotient ring Q, then $T(M)$ is the kernel of the map $m\longrightarrow m\otimes_R 1$ of M_R into $M\otimes_R Q$.*

Also Jain, Mohamed and Singh ([2], Theorem 2, 3, p. 74) verify the following.

PROPOSITION 2. *The following conditions are equivalent.*

(1) *R is a q-ring,*

(2) *R is right self-injective, and every right ideal of R is the form eI, where e is an idempotent in R, and I is a two-sided ideal in R.*

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(3) *R is right self-injective, and every large right ideal of R is two-sided.*

Let M_R be any R-module. In [1], Eckmann and Schopf have shown that the minimal R-injective extension always exists and this coincides with the maximal R-essential extension and equals to R-injective, essential extension of M_R.

In this case $E(M_R)$ denoted the minimal injective extension of M_R and is called the injective hull of M_R. And particularly $E(R)$ denoted the injective hull of R_R.

In general the injective hull $E(R)$ of R_R doesn't have ring structure. But under the appropriate condition, the R-module $E(R)$ can have ring structure. The following states the condition of R so that $E(R)$ has ring structure having R as a subring.

THEOREM 3. *If R satisfies the condition $T(E(R)/R)=E(R)/R$, then we can give a ring structure to $E(R)$ having R as a subring.*

Proof. At first we show that $\mathrm{Hom}_R(E(R)/R, E(R))=0$. If there exists f in $\mathrm{Hom}_R(E(R)/R, E(R))$ such that $\mathrm{Im}(f)$ is a non-zero R-submodule of $E(R)$. Then $\mathrm{Im}(f)\cap R\neq 0$ because $E(R)$ is an R-essential extension of R. Now take a nonzero element $a\in \mathrm{Im}(f)\cap R$, $a=f(\bar{c})$ for some $\bar{c}=c+R\in E(R)/R$. Since $T(E(R)/R)=E(R)/R$, there exists a regular element b in R such that $\bar{c}b=(c+R)b=cb+R=0$ i.e., $cb\in R$. It follows $a=0$ from the fact $f(\bar{c}b)=f(\bar{c})b=ab=0$ and b is regular in R.

Now form the short exact sequence

$$0\longrightarrow R\longrightarrow E(R)\longrightarrow E(R)/R\longrightarrow 0, \tag{1}$$

And apply the left exact functor $\mathrm{Hom}_R(R, E(R))$ to (1), we obtain the following exact sequence (2) from the fact $\mathrm{Hom}_R(E(R)/R, E(R))=0$ such that

$$0\longrightarrow \mathrm{Hom}_R(E(R), E(R))\longrightarrow \mathrm{Hom}_R(R, E(R))\longrightarrow 0 \tag{2}$$

i.e., $\mathrm{Hom}_R(E(R), E(R))\simeq \mathrm{Hom}_R(R, E(R))$.

Since R has identity, we can identify $E(R)$ and $\mathrm{Hom}_R(R, E(R))$ as an additive group structure naturally under the mapping a in $E(R)$ to a_L in $\mathrm{Hom}_R(R, E(R))$. And now a_L has the unique extension ϕ_a in $\mathrm{Hom}_R(E(R), E(R))$ by (2). To define the multiplication $\circ$ on $E(R)$, let a, b be elements of $E(R)$. And define such as $a\circ b=\phi_a(b)$, then $E(R)$ has a ring structure having R as a subring. Thus the proof is completed.

3. Characterization.

The above theorem shows that the injective hull $E(R)$ has a ring structure under the condition $T(E/(R)/R)=E(R)/R$. From this fact we can deduce the following main result which characterizes right orders in q-rings.

THEOREM 4. *A necessary and sufficient condition that a given ring R can be embedded in a q-ring Q as a right order is that R satisfies the following conditions:*

(1) $T(E(R)/R)=E(R)/R$,

(2) *for any regular b in R, $bE(R)\supseteq R$,*

(3) *every large right ideal of R which has the form $L\cap R$ where L is a large right ideal of $E(R)$ is two-sided,*

(4) *for every large right ideal L of $E(R)$ and every regular b in R, $L\subseteq bL$.*

Proof. Assume that R is a right order in a q-ring Q. If we show $E(R)=Q$, then the condition (1) and (2) are easily verified because each regular element in R annihilates only 0 in Q.

Since $Q=\{ab^{-1}|a\in R,\ b \text{ regular in } R\}$, Q_R is an essential extension of R_R which mentioned already. To show $Q=E(R)$, it remains to prove that the overring Q of R is an R-injective module. Let A and B be R-modules such that $B\subseteq A$ and $f: B\to Q$ be an R-module homomorphism, then f can be shifted to $\bar{f}: B/\mathrm{Ker}(f)\to Q$ which is a monomorphism. Thus we can assume $f: B\to Q$ as an R-module monomorphism at first without loss of generality.

If $f: B\to Q$ is an R-module monomorphism, then $T(B)=T(\mathrm{Im}\, f)=0$ because $T(\mathrm{Im} f)\cap R=0$ and Q is an essential extension of R. Therefore we can conclude the morphism $B\to B\otimes_R Q$ which sending b to $b\otimes 1$ is an R-monomorphism by Proposition 1. And the given R-monomorphism $f: B\to Q$ induced a Q-homomorphism $f\otimes 1: B\otimes_R Q\to Q\otimes_R Q$. Since ${}_RQ$ is a flat module, $B\otimes_R Q\to Q\otimes_R Q$ is a Q-monomorphism by the the left exactness of the functor $\otimes_R Q$. Thus we get a Q-homomorphism $\bar{f}: A\otimes_R Q\to Q\otimes_R Q$ such that each rectangle of the following diagram commutes by the right self-injectivity of Q.

$$\begin{array}{ccccc} B_R & \longrightarrow & (B\otimes_R Q)_Q & \longrightarrow & (A\otimes_R Q)_Q \\ {\scriptstyle f}\downarrow & & {\scriptstyle f\otimes 1}\downarrow & & \vdots{\scriptstyle \bar{f}} \\ Q_R & \longrightarrow & (Q\otimes_R Q)_Q & \xleftrightarrow{\text{isomorphism}} & Q_Q \end{array}$$

This gives an R-homomorphism f' of A into Q via the map $\bar{f}$. This f' is an R-extension of f to A. Therefore Q_R is R-injective. Thus $Q=E(R)$. From this fact $Q=E(R)$, the condition (1) and (2) are verified immediately. To show the condition (3), let L be a right ideal of $E(R)=Q$ such that $L\cap R$ is large in R. Since Q is a q-ring, the given large right ideal L is two-sided in Q by Proposition 2. And since $(L\cap R)Q=L$, $R(L\cap R)\subseteq R(L\cap R)Q=RL\subseteq L$ i.e., $L\cap R$ is a left ideal in R. The condition (4) is clear. Conversely assume the given condition (1), (2), (3) and (4). For any a in R, the left multiplication a_L is an element of $\mathrm{Hom}_R(R, E(R))\simeq\mathrm{Hom}_R(E(R), E(R))\simeq E(R)$. And the map $a\to a_L$ is a ring monomorphism of R into $E(R)$.

Now let b be a regular in R, then b_L is an automorphism of $E(R)_R$. For b_L has zero kernel because if b_L has non-zero kernel, it's restriction to R has also non-zero kernel. And by the condition (2), since $b_L E(R)$ is an R-injective module containing R, $b_L(E(R))\supseteq E(R)$. This means that b is an automorphism of $E(R)$. In this case let ϕ be the inverse of b_L, then $\phi(1)$ is the inverse of b. Now let q be any element in $E(R)$, then there exists a regular b in R such that $qb=a\in R$ exactly by the condition (1). It follows that $q=ab^{-1}$ and we have shown that R is a right order in the ring $E(R)$.

Since $E(R)=Q$ is a ring with identity, to show $E(R)=Q$ is right self-injective it is sufficient to prove that every Q-homomorphism from I to Q is a left multiplication by an element of Q, where I is a right ideal of Q, i.e., Q satisfies the Baer's condition. Now let $f\in\mathrm{Hom}_Q(I, Q)$, where I is a right ideal of Q. Then $f_0=f|I\cap R$ belongs to $\mathrm{Hom}_R(I\cap R, Q)$. The R-injectivity of $Q=E(R)$ implies that $f_0(x)=ax$ for some a in R. And $I=(I\cap R)Q$, i.e., $I=(I\cap R)Q=\{ab^{-1}|a\in I\cap R,\ b \text{ regular in } R\}$. For every element

$i \in I$, i has the form xy^{-1}, $x \in I \cap R$, y regular in R. Therefore we have $f(i)=f(xy^{-1})$ $=f(x)y^{-1}=f_0(x)y^{-1}=(ax)y^{-1}=a(xy^{-1})=ai$, i. e., Q_R satisfies the Baer's condition.

On the other hand to show $E(R)=Q$ is a q-ring, by Proposition 2, it remains to prove that every large right ideal of Q is two-sided. Let L be a large right ideal of Q, then $L \cap R$ is also a large right ideal of R. For let I be a right ideal of R such that $(L \cap R) \cap I=0$, then $L \cap IQ=0$. This fact follows from that IQ is the right ideal of Q which is generated by I. And $IQ=\{xy^{-1} | x \in I,\ y \text{ regula in } R\}$. Therefore we have $I=0$, and this means that $L \cap R$ is large in R. Since $L \cap R$ is two-sided in R by the condition (3), we have $(L \cap R)R \subseteq L \cap R$ and $R(L \cap R) \subseteq L \cap R$. And since $L=(L \cap R)Q$, RL $=R(L \cap R)Q \subseteq (L \cap R)Q=L$. Now to prove $QL=Q((L \cap R)Q) \subseteq L$, for any $q \in Q$, $q=a$ b^{-1} and $r \in L$ such that $r=cd^{-1}$, $c \in L \cap R$, then $qr=(ab^{-1})(cd^{-1})=(ac')(db')^{-1}$, where $cb'=bc'$, $c', b' \in R$, b' regular in R. Since $cb'=bc'$, $c'=b^{-1}cb' \in L$ by the condition (4) and the fact $c \in L \cap R$. The fact $RL \subseteq L$ implies $ac' \in L$. Therefore $qr=(ac')(db')^{-1} \in$ L because db' is regular in R and $ac' \in L \cap R$, i. e., $QL \subseteq L$. Thus we have the fact that Q is a q-ring and now the proof is completed.

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