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THE MINIMAL PROJECTIVE EXTENSION OF THE STRUCTURE SPACE OF A RING

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The projectivity underlying question has been introduced in [3] and a great deal of work has been done by [1, 2, 4, 6 and 7]. The purpose of this note is to present a new relevant result.

In the sequel, we will let $\mathbf{C}$ denote the category of Hausdorff spaces and their continuous maps and $\mathbf{K}$ denote the category of compact Hausdorff spaces and their continuous maps. By a projective object in the category $\mathbf{C}$, we mean an object $X \in \mathbf{C}$ such that, if $h : X \to Z$ in $\mathbf{C}$ is onto, then any mapping $f : X \to Z$ can be lifted; that is, there is a mapping $g : X \to Y$ such that $hg = f$. By a projective cover of X in the category $\mathbf{C}$ we mean an onto mapping $f : K \to X$ such that K is a projective object in $\mathbf{C}$ and f maps no proper closed subset of K onto X. For a compact Hausdorff space E, if $f : K \to E$ is the projective cover of E, K is called the minimal projective extension of E. Amongst other things [3] and [7] have proved the following.

THEOREM 1. *In the category* $\mathbf{K}$, *the projective objects are precisely the extremally disconnected spaces.*

THEOREM 2. *For every object* $X \in \mathbf{K}$, *there exists a minimal projective extension* P *in* $\mathbf{K}$ *of* X, *and if* P_1, P_2 *are minimal projective extensions of* X, *then* P_1 *and* P_2 *are homeomorphic.*

With the same technique applied to the proof of the Proposition 3 in [1] one can show the following:

Let E be a compact Hausdorff space, let $R_0(E)$ be the collection of its regular open subsets and $\Omega(E)$ be the space of maximal filters $M \subseteq R_0(E)$ whose topology is generated by the set $\Omega_v(E) = \{M \mid v \in M \in \Omega(E)\}$. The $\Omega(E)$ is an extremally disconnected compact Hausdorff space. Denote the mapping $\Omega(E) \to E$ which assigns to each $M \in \Omega(E)$ its limit point by $\lim_E$. Then $\lim_E$: $\Omega(E) \to E$ is the projective cover of E, i. e., $\Omega(E)$ is the minimal projective extension of E.

In what follows, every ring will mean a commutative semi-simple ring with unity.

Let A be a ring, and denote by $J(A)$ the set of all annihilar ideals of A and by $\mathbf{M}(A)$ the set of all maximal ideals of A. Define, for each $a \in A$, the function $\hat{a}$ on $\mathbf{M}(A)$ by $\hat{a}(M) = a + M \in A/M$. Let $S(\hat{a})$ be the cozero set of $\hat{a}$, i. e. $S(\hat{a}) = \{M \mid \hat{a}(M) \neq 0\}$. The topology on $\mathbf{M}(A)$ generated by the cozero sets $\{S(\hat{a})\}_{a \in A}$ will be referred to as the Stone-Zariski topology.

In [6] it is shown that the structure space of the maximal ring of quotients of $C(X)$

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endowed with the Stone-Zariski topology is the minimal projective extension of βX. It will be seen that one can generalize this result for the class of structure spaces of commutative semi-simple rings. To proceed we introduce some fundamental notations. For an ideal I of A define $\mathsf{M}(I)$ as $\mathsf{M}(I)=\{M|M\in\mathsf{M}(A), M\not\supset I\}$, for a subset U of $\mathsf{M}(A)$ define $\Delta(U)$ as $\Delta(U)=\bigcap\{M\in U\}$ and for a subset U of a topological space E define U^1 as $U^1=E-Cl_E U$ where Cl_E denotes the closure operator with respect to the space E. Then one can show that V is a regular open set of E if and only if $V=U^1$ for some open set U of E. Also noting that $Cl_{\mathsf{M}(A)}U=\{M|M\in\mathsf{M}(A), M\supset\Delta(U)\}$ one shows that $U^1=(\mathsf{M}\circ\Delta)(U)$ and $I^*=(\Delta\circ\mathsf{M})(I)$ where $I^*=\{a|aI=0\}$.

These preceding results lead to the formation of a lemma and a theorem due to [5], namely

LEMMA 3. $J(A)\cong R_0(\mathsf{M}(A))$.

THEOREM 4. $J(A)\cong J(Q(A))$, *where* $Q(A)$ *denotes the maximal ring of quotients of* A. *If* A *is rationally complete, then* $J(A)\cong A^0$, A^0 *denotes the algebra of idempotents of* A.

As a preliminary step to obtaining the final result, we state the following lemma.

LEMMA 5. *If* A *is regular, then* $\mathsf{M}(A)\cong\mathsf{M}(A^0)$.

Proof. Clearly the mapping $M\to M\cap A^0$ is one-to-one and onto. For each $e\in A^0$, put $\mathsf{M}^0(e)=\{p\in\mathsf{M}(A^0)|p\not\ni e\}$. Then $\{\mathsf{M}^0(e)\}_{(e\in A^0)}$ form a base for $\mathsf{M}(A^0)$. Noting that for any $a\in A$, there exists $r\in A$ such that $ar\in A^0$. Now one shows that $\mathsf{M}(A)\cap A^0=\mathsf{M}^0(ar)$ for $a\in A$ and $\mathsf{M}(e)\cap A^0=\mathsf{M}^0(e)$ for each $e\in A^0$. This shows that the mapping carries the base for $\mathsf{M}(A)$ onto the base for $\mathsf{M}(A^0)$.

THEOREM 6. *If* $\mathsf{M}(A)$ *is Hausdorff, then the structure space* $\mathsf{M}(Q(A))$ *is the minimal projective extension of* $\mathsf{M}(A)$.

Proof. Since $Q(A)$ is regular, we have $\mathsf{M}(Q(A))\cong\mathsf{M}(Q(A)^0)$ and $Q(A)^0\cong J(Q(A))$ $\cong R_0(\mathsf{M}(A))$. Now, let $\Omega(\mathsf{M}(A))$ be the space of maximal filters U in $R_0(\mathsf{M}(A))$. Then $\mathsf{M}[R_0(\mathsf{M}(A))]\cong\Omega(\mathsf{M}(A))$ under the mapping $M\to\mathsf{U}_M$ where $\mathsf{U}_M\equiv\{p|p'\in M$, p' is the complement of P w.r.t. $\mathsf{M}(A)\}$. Since $\mathsf{M}(A)$ is Hausdorff, the mapping $\lim_{M(A)}$: $\Omega(\mathsf{M}(A))\to\mathsf{M}(A)$ is the projective cover of $\mathsf{M}(A)$.

COROLLARY 1: *For a compact Hausdorff space* E, $\mathsf{M}(Q(C(E)))$ *is the minimal projective extension of* E.

The discussion up to now has been within the framework of the injective system. Within the projective system as dual, we also see the following:

Let X be a completely regular Hausdorff space, and σ_0 denote the collection of dense open subsets of X. For each $D\in\sigma_0$, βD denotes its Stone-Čech compactification and $C^*(D)$ denotes the ring of real-valued bounded continuous functions on D. Then the family $\{\beta D\}$ $(D\in\sigma_0)$ together with $\{\phi_{DE}^*\}_{E\subset D,\ E,D\in\sigma}$ where ϕ_{DE}^* is defined as the dual of the natural restritcion homomorphism ϕ_{DE}: $C^*(D)\to C^*(E)$ form a projective system in the category of compact Hausdorff spaces and their continuous maps, and one can show that the system has the project limit $\lim_{\leftarrow D \in \mathscr{D}} \beta D$.

COROLLARY 2. *The projective limit* $\lim_{\leftarrow D \in \mathscr{D}} \beta D$ *is the minimal projective extension of* βX.

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