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COMPLETE SUBMANIFOLDS OF CODIMENSION 2 IN AN EVEN-DIMENSIONAL EUCLIDEAN SPACE

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Dedicated to Professor Kentaro Yano on his sixtieth birthday

§0. Introduction.

Recently, D.E. Blair [1], G.D. Ludden [1], M. Okumura [13] and K. Yano [1], [13] found that a submanifold of codimension 2 in an almost Hermitian manifold or a hypersurface of an almost contact manifold admits what we call an (f, g, u, v, λ)-structure. The structure induced on the former submanifold has been studied by U-Hang Ki [4], [5], [12], M. Okumura [6], [14] and K. Yano [12], [14] and the structure induced on the latter hypersurface has been studied by D.E. Blair [2], S.S. Eum [8], S. Ishihara [10], U-Hang Ki [8], G.D. Ludden [2] and K. Yano [2], [8], [10].

Moreover, submanifolds of codimension 2 in an even dimensional Euclidean space have been studied by K. Yano and one of the present authors and the main theorem related to the (f, g, u, v, λ)-structure in [12] is as follows.

THEOREM A. *Let M^{2n} be a complete submanifold of codimension 2 in a $(2n+2)$-dimensional Euclidean space E^{2n+2} such that (1) the scalar curvature of M^{2n} is constant and (2) there are global unit normals C and D to M^{2n} which are parallel in the normal bundle. If $fH=Hf$ and $fK=-Kf$ hold, where H and K are the second fundamental tensors of M^{2n} with respect to C and D respectively, f being the tensor field of type $(1,1)$ appearing in the (f, g, u, v, λ)-structure induced on M^{2n},* then M^{2n} *is in E^{2n+2}, provided that $\lambda(1-\lambda^2)$ is non-zero almost every-where in M^{2n}, congruent to one of the following submanifolds:*

$$E^{2n}, \quad S^{2n}, \quad S^n(r)\times S^n(r), \quad S^l(r)\times E^{2n-l} \quad (l=1,2,\cdots,2n-1),$$

$$S^k(r)\times S^k(r)\times E^{2n-2k} \quad (k=1,2,\cdots,n-1),$$

where $S^k(r)$ denotes a k-dimensional sphere of radius $r(>0)$ imbedded naturally in E^{2n+2}.

The purpose of the present paper is to obtain the same conclusion as in Theorem A by replacing the condition (1) or (2) assumed in Theorem A with any another condition.

§1. Submanifolds of codimension 2 in an even-dimensional Euclidean space.

Let E^{2n+2} be a $(2n+2)$-dimensional Euclidean space and X the position vector starting from the origin of E^{2n+2} and ending at a point of E^{2n+2}. The E^{2n+2} being even-dimensional, it can be regarded as a flat Kaehlerian manifold with the numerical structure tensor $F: F^2=-I$, where I denotes the unit tensor and $FY\cdot FZ=Y\cdot Z$ for arbitrary

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vector fields Y and Z, where the dot denotes the inner product of vectors of E^{2n+2}.

Let M^{2n} be a $2n$-dimensional orientable manifold covered by a system of coordinate neighborhoods $\{U, x^h\}$ where here and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, 2n\}$ and the summation convention will be used with respect to these indices.

We assume that M^{2n} is immersed in E^{2n+2} by $X: M^{2n} \longrightarrow E^{2n+2}$ and put $X_i = \partial_i X$, $\partial_i = \partial/\partial x^i$. Then X_i are $2n$ linearly independent local vector fields tangent to $X(M^{2n})$ and $g_{ji} = X_j \cdot X_i$ are local components of the tensor representating the Riemannian metric induced on M^{2n} from that of E^{2n+2}.

We assume that we can take two globally defined mutually orthogonal unit normals C and D to $X(M^{2n})$ in such a way that $X_1, X_2, \cdots, X_{2n}, C, D$ give the positive orientation of E^{2n+2}. In the sequel we identify $X(M^{2n})$ with M^{2n} itself.

The transforms FX_i by F can be expressed as linear combinations of X_h, C and D, that is, we have equations of the form

$$FX_i = f_i{}^h X_h + u_i C + v_i D, \tag{1.1}$$

where $f_i{}^h$ are components of a tensor field of type (1, 1) and u_i, v_i are those of 1-forms of M^{2n}, all globally defined on M^{2n}. The transforms FC and FD of C and D by F can be expressed as

$$FC = -u^h X_h + \lambda D, \tag{1.2}$$

$$FD = -v^h X_h - \lambda C \tag{1.3}$$

respectively, where $u^h = u_i g^{ih}$, $v^h = v_i g^{ih}$ and λ is a function on M^{2n}, because

$$X_i \cdot FC = FX_i \cdot F^2 C = -FX_i \cdot C = -u_i,$$

$$X_i \cdot FD = FX_i \cdot F^2 D = -FX_i \cdot D = -v_i,$$

$$FC \cdot D = F^2 C \cdot FD = -C \cdot FD.$$

Applying F to (1.1), (1.2) and (1.3) and using $F^2 = -I$, (1.1), (1.2) and (1.3), we find (cf. [13])

$$\begin{aligned} &f_i{}^t f_t{}^h = -\delta_i^h + u_i u^h + v_i v^h, \\ &u_t f_i{}^t = +\lambda v_i, \qquad f_t{}^h u^t = -\lambda v^h, \\ &v_t f_i{}^t = -\lambda u_i, \qquad f_t{}^h v^t = +\lambda u^h, \\ &u_i u^i = v_i v^i = 1 - \lambda^2, \qquad u_i v^i = 0. \end{aligned} \tag{1.4}$$

We also have, from (1.1),

$$g_{ts} f_j{}^t f_i{}^s = g_{ji} - u_j u_i - v_j v_i \tag{1.5}$$

by virtue of $FX_j \cdot FX_i = X_j \cdot X_i = g_{ji}$. We can easily see that $f_{ji} = f_j{}^t g_{ti}$ is skew-symmetric in lower indices j and i.

The structure defined on M^{2n} by such a set of a tensor field f of type (1, 1), a Riemannian metric g, two 1-forms u and v and a function λ satisfying (1.4) and (1.5) is called an (f, g, u, v, λ)-structure (cf. [13]).

We denote by $\{_j{}^h{}_i\}$ the Christoffel symbols formed with g_{ji} and by ∇_i the operator of covariant differentiation with respect to $\{_j{}^h{}_i\}$. Then equations of Gauss are

$$\nabla_j X_i = \partial_j X_i - \{_j{}^h{}_i\} X_h = h_{ji} C + k_{ji} D, \tag{1.6}$$

where h_{ji} and k_{ji} are components of the second fundamental tensors with respect to C and D respectively, and equations of Weingarten are

$$\begin{aligned} \nabla_j C &= \partial_j C = -h_j{}^h X_h + l_j D, \\ \nabla_j D &= \partial_j D = -k_j{}^h X_h - l_j C, \end{aligned} \tag{1.7}$$

where $h_j{}^h$ and $k_j{}^h$ are given respectively by $h_j{}^h = h_{ji} g^{ih}$ and $k_j{}^h = k_{ji} g^{ih}$, and l_j are components of the third fundamental tensor, i. e., components of the connection induced on the normal bundle.

Now, differentiating (1.1) covariantly and taking account of $\nabla_j F = 0$ and of equations of Gauss and Weingarten, we obtain (cf. [5], [14])

$$\nabla_j f_i{}^h = -h_{ji} u^h + h_j{}^h u_i - k_{ji} v^h + k_j{}^h v_i, \tag{1.8}$$

$$\nabla_j u_i = -h_{jt} f_i{}^t - \lambda k_{ji} + l_j v_i, \tag{1.9}$$

$$\nabla_j v_i = -k_{jt} f_i{}^t + \lambda h_{ji} - l_j u_i. \tag{1.10}$$

Similarly, we have, from (1.2),

$$\nabla_j \lambda = -h_{ji} v^i + k_{ji} u^i. \tag{1.11}$$

In the sequel, we need the structure equations of the submanifold M^{2n}, that is, the following equations of Gauss

$$K_{kjih} = h_{kh} h_{ji} - h_{jh} h_{ki} + k_{kh} k_{ji} - k_{jh} k_{ki}, \tag{1.12}$$

where K_{kjih} are covariant components of the curvature tensor of M^{2n}, and equations of Codazzi and Ricci

$$\nabla_k h_{ji} - \nabla_j h_{ki} - l_k k_{ji} + l_j k_{ki} = 0, \tag{1.13}$$

$$\nabla_k k_{ji} - \nabla_j k_{ki} + l_k h_{ji} - l_j h_{ki} = 0, \tag{1.14}$$

$$\nabla_j l_i - \nabla_i l_j + h_{jt} k_i{}^t - h_{it} k_j{}^t = 0. \tag{1.15}$$

§2. The case in which f and H commute and f and K anticommute.

We suppose that f and H commute, i. e.,

$$f_j{}^t h_t{}^h - h_j{}^t f_t{}^h = 0, \tag{2.1}$$

which is equivalent to

(2.2) $$h_{jt}f_i^t+h_{it}f_j^t=0,$$

that is, $h_{jt}f_i^t$ is skew-symmetric. We suppose also that f and K anticommute, i.e.,

(2.3) $$f_j^tk_t^h+k_j^tf_t^h=0,$$

which is equivalent to

(2.4) $$k_{jt}f_i^t-k_{it}f_j^t=0,$$

that is, $k_{jt}f_i^t$ is symmetric.

K. Yano and one of the present authors proved in [12]

PROPOSITION 2.1. *Let $X(M^{2n})$ be a submanifold of codimension 2 in a $(2n+2)$-dimensional Euclidean space E^{2n+2} such that (2.1) and (2.3) are satisfied and the function $\lambda(1-\lambda^2)$ is almost everywhere non-zero in M^{2n}. Then*

(2.5) $$k_t^t=0,$$

(2.6) $$h_{ji}u^i=pu_j, \qquad h_{ji}v^i=pv_j,$$

(2.7) $$k_{ji}u^i=\alpha u_j+\beta v_j, \qquad k_{ji}v^i=\beta u_j-\alpha v_j,$$

p, α and β being given respectively by

$$p=\frac{h_{ts}u^tu^s}{1-\lambda^2}=\frac{h_{ts}v^sv^t}{1-\lambda^2}, \qquad \alpha=\frac{k_{ts}u^tu^s}{1-\lambda^2}=\frac{k_{ts}v^tv^s}{1-\lambda^2}, \qquad \beta=\frac{k_{ts}u^tv^s}{1-\lambda^2}.$$

PROPOSITION 2.2. *Under the same assumptions as in Proposition 2.1, we have*

(2.8) $$h_j^tk_{th}-k_j^th_{th}=0.$$

Proof. Differentiating the second equation of (2.6) covariantly. we find

$$(\nabla_jh_i^t)v_t+h_i^t\nabla_jv_t=(\nabla_jp)v_i+p\nabla_jv_i,$$

from which, using (1.10),

$$(\nabla_jh_i^t)v_t+h_i^t(-k_{js}f_t^s+\lambda h_{jt}-l_ju_t)=(\nabla_jp)v_i+p(-k_{jt}f_i^t+\lambda h_{ji}-l_ju_i).$$

Taking the skew-symmetric part with respect to j and i and using equation (1.13) of Codazzi, we obtain

$$(l_jk_i^t-l_ik_j^t)v_t+(h_j^tk_i^s-h_i^tk_j^s)f_{ts}-l_jh_i^tu_t+l_ih_j^tu_t$$
$$=(\nabla_jp)v_i-(\nabla_ip)v_j-p(l_ju_i-l_iu_j),$$

from which, using (2.6) and (2.7),

$$l_j(\beta u_i-\alpha v_i)-l_i(\beta u_j-\alpha v_j)+(h_j{}^t k_i{}^s-h_i{}^t k_j{}^s)f_{ts}-p(l_j u_i-l_i u_j)$$
$$=(\nabla_j p)v_i-(\nabla_i p)v_j-p(l_j u_i-l_i u_j),$$

or

$$(2.9)\qquad (h_j{}^t k_i{}^s-h_i{}^t k_j{}^s)f_{ts}=(\nabla_j p+\alpha l_j)v_i-(\nabla_i p+\alpha l_i)v_j-\beta(l_j u_i-l_i u_j).$$

Transvecting (2.9) with u^i and using (2.6) and (2.7), we find

$$[h_j{}^t(\alpha u^s+\beta v^s)-pu^t k_j{}^s]f_{ts}=-(u^i\nabla_i p+\alpha l_i u^i)v_j-\beta l_j(1-\lambda^2)+\beta(l_i u^i)u_j,$$

or

$$(2.10)\qquad \beta(1-\lambda^2)l_j=\beta(l_i u^i)u_j-(u^i\nabla_i p+\alpha l_i u^i)v_j.$$

On the other hand, transvecting (2.9) with v^i and using (2.6) and (2.7), we find

$$[h_j{}^t(\beta u^s-\alpha v^s)-pv^t k_j{}^s]f_{ts}=(1-\lambda^2)(\nabla_j p+\alpha l_j)-(v^i\nabla_i p+\alpha l_i v^i)v_j+\beta(l_i v^i)u_j,$$

or

$$(2.11)\qquad (1-\lambda^2)(\nabla_j p+\alpha l_j)=-\beta(l_i v^i)u_j+(v^i\nabla_i p+\alpha l_i v^i)v_j.$$

Thus, multiplying (2.9) by $1-\lambda^2$ and using (2.10) and (2.11), we obtain

$$(1-\lambda^2)(h_j{}^t k_i{}^s-h_i{}^t k_j{}^s)f_{ts}=-[u^t\nabla_t p+\alpha l_t u^t+\beta l_t v^t](u_j v_i-u_i v_j).$$

Transvecting this with u^i, we find

$$u^t\nabla_t p+\alpha l_t u^t+\beta l_t v^t=0,$$

and consequently we have

$$(h_j{}^t k_i{}^s-h_i{}^t k_j{}^s)f_{ts}=0,$$

from which, using (2.2) and (2.4)

$$h_j{}^t k_t{}^s f_{is}+h_s{}^t k_j{}^s f_{ti}=0.$$

Transvecting this with $f_h{}^i$ and using (1.4), we find (2.8). Thus Propositions 2.2 is proved.

PROPOSITION 2.3. *Under the same assumptions as in Proposition* 2.1, *we have*

$$(2.12)\qquad h_{jt}h_h{}^t=ph_{jh},\qquad or\qquad h_t{}^h h_i{}^t=ph_i{}^h.$$

Proof. Differentiating the first equation of (2.6) covariantly and using (1.9), we find

$$(\nabla_j h_i{}^t)u_t+h_i{}^t(-h_{js}f_t{}^s-\lambda k_{jt}+l_j v_t)=(\nabla_j p)u_i+p(-h_{jt}f_i{}^t-\lambda k_{ji}+l_j v_i),$$

from which, using (1.13), (2.6), (2.7) and (2.8),

$$(2.13)\qquad 2h_j{}^t h_i{}^s f_{ts}=(\nabla_j p-\alpha l_j)u_i-(\nabla_i p-\alpha l_i)u_j-\beta l_j v_i+\beta l_i v_j-2ph_{jt}f_i{}^t.$$

Transvecting (2.13) with u^i and v^i, we find respectively

(2.14) $$(1-\lambda^2)(\nabla_j p-\alpha l_j)=u^t(\nabla_t p-\alpha l_t)u_j-\beta(l_t u^t)v_j,$$

(2.15) $$(1-\lambda^2)\beta l_j=-v^t(\Delta_t p-\alpha l_t)u_j+\beta(l_t u^t)v_j.$$

Substituting (2.14) and (2.15) into (2.13), we find

(2.16) $$2(1-\lambda^2)h_j{}^t h_i{}^s f_{ts}=(\beta l_t u^t+v^t\nabla_t p-\alpha l_t v^t)(v_i u_j-v_j u_i)-2p(1-\lambda^2)h_{jt}f_i{}^t,$$

from which, transvecting with u^i,

$$\beta l_t u^t+v^t\nabla_t p-\alpha l_t v^t=0.$$

Thus (2.16) becomes

$$h_j{}^t h_i{}^s f_{st}=p h_{jt} f_i{}^t, \qquad \text{or} \qquad h_j{}^t h_t{}^s f_{is}=p h_{jt} f_i{}^t,$$

from which, transvecting with $f_h{}^i$, we find

$$h_j{}^t h_t{}^s(-g_{sh}+u_s u_h+v_s v_h)=p h_{jt}(-\delta_h^t+u_h u^t+v_h v^t).$$

Therefore, using (2.6), wh have (2.12). This completes the proof of Proposition 2.3.

PROPOSITION 2.4. *Under the same assumptions as in Proposition* 2.1, *we have*

(2.17) $$(1-\lambda^2)(k_j{}^t k_{ti}+\beta h_{ji})=[\alpha^2+\beta(\beta+p)](u_j u_i+v_j v_i).$$

Proof. Differentiating the second equation of (2.7) covariantly, we find

$$(\nabla_j k_i{}^t)v_t+k_i{}^t(-k_{js}f_t{}^s+\lambda h_{jt}-l_j u_t)$$
$$=(\nabla_j\beta)u_i+\beta(-h_{jt}f_i{}^t-\lambda k_{ji}+l_j v_i)-(\nabla_j\alpha)v_i-\alpha(-k_{jt}f_i{}^t+\lambda h_{ji}-l_j u_i).$$

Taking the skew-symmetric part and using (1.14), (2.6) and (2.7), we find

(2.18) $$2k_j{}^t k_i{}^s f_{ts}=(\nabla_j\beta+2\alpha l_j)u_i-(\nabla_i\beta+2\alpha l_i)u_j-(\nabla_j\alpha-p l_j-2\beta l_j)v_i$$
$$+(\nabla_i\alpha-p l_i-2\beta l_i)v_j-2\beta h_{jt}f_i{}^t,$$

from which, transvecting u^i and v^i,

(2.19) $$(1-\lambda^2)(\nabla_j\beta+2\alpha l_j)=Au_j+Bv_j,$$

(2.20) $$(1-\lambda^2)(\nabla_j\alpha-pl_j-2\beta l_j)=Cu_j+Dv_j,$$

where, we have put

(2.21) $$A=u^t\nabla_t\beta+2\alpha u^t l_t, \qquad B=v^t\nabla_t\beta+2\alpha v^t l_t,$$
$$C=u^t\nabla_t\alpha-(p+2\beta)u^t l_t, \qquad D=v^t\nabla_t\alpha-(p+2\beta)v^t l_t.$$

Multiplying (2.18) by $(1-\lambda^2)$ and substituting (2.19), (2.20) and (2.21) into the equation obtained, we find

(2.22) $$2(1-\lambda^2)k_j{}^t k_i{}^s f_{ts}=[u^t\nabla_t\alpha+v^t\nabla_t\beta+2\alpha v^t l_t-(p+2\beta)l_t u^t](u_i v_j-u_j v_i)$$
$$-2(1-\lambda^2)\beta h_{jt}f_i{}^t,$$

from which, transvecting with $u^j v^i$ and using (2.6) and (2.7),

(2.23) $$u^t\nabla_t\alpha+v^t\nabla_t\beta+2\alpha v^t l_t-(p+2\beta)u^t l_t=-2\lambda[\alpha^2+\beta(\beta+p)].$$

Thus (2.22) becomes

$$(1-\lambda^2)k_j{}^t k_t{}^s f_{is}=\lambda[\alpha^2+\beta(\beta+p)](u_j v_i-u_i v_j)-(1-\lambda^2)\beta h_{jt}f_i{}^t.$$

Transvecting this with $f_h{}^i$, we find

$$(1-\lambda^2)k_j{}^t k_t{}^s(-g_{sh}+u_s u_h+v_s v_h)$$
$$=-\lambda^2[\alpha^2+\beta(\beta+p)](u_j u_h+v_j v_h)-(1-\lambda^2)\beta h_{jt}(-\delta_h{}^t+u_h u^t+v_h v^t),$$

from which, using (2.6) and (2.7),

$$(1-\lambda^2)(k_j{}^t k_{th}+\beta h_{jh})=[\alpha^2+\beta(\beta+p)](u_j u_h+v_j v_h),$$

which proves Proposition 2.4.

We have from equation (1.12) of Gauss,

$$K_{ji}=h_t{}^t h_{ji}-h_{jt}h_i{}^t-k_{jt}k_i{}^t,$$

from which, using (2.12),

$$K_{ji}=(h_t{}^t-p)h_{ji}-k_{jt}k_i{}^t.$$

From this and (2,17), we find

$$(1-\lambda^2)K_{ji}=(1-\lambda^2)(h_t{}^t-p+\beta)h_{ji}-[\alpha^2+\beta(\beta+p)](u_j u_i+v_j v_i),$$

from which, transvecting with g^{ji},

(2.24) $$g^{ji}K_{ji}=(h_t{}^t-p+\beta)h_t{}^t-2[\alpha^2+\beta(\beta+p)],$$

which gives the scalar curvature of M^{2n}.

§3. Complete submanifolds with certain conditions

Let M^{2n} be a submanifold of codimension 2 in a $(2n+2)$-dimensional Euclidean space E^{2n+2} such that (2.1) and (2.3) are satisfied and the function $\lambda(1-\lambda^2)$ is non-zero almost everywhere in M^{2n}. Then the mean curvature vector of M^{2n} is defined to be

$$W=\frac{1}{2n}(h_t{}^t C+k_t{}^t D),$$

from which, using (2.5),

(3.1) $$W=\frac{1}{2n}h_t{}^tC.$$

Differentiating (3.1) covariantly and making use of (1.7), we find

(3.2) $$2n\nabla_j W=(\nabla_j h_t{}^t)C+h_t{}^tl_jD-h_t{}^th_j{}^sX_s.$$

We here assume that the submanifold M^{2n} is complete, the scalar curvature $g^{ji}K_{ji}$ of M^{2n} is constant and the mean curvature vector W of M^{2n} is parallel in the normal bundle. Then we see from (3.2) that $h_t{}^t$ is constant and $h_t{}^tl_j=0$, from which

(3.3) $$h_t{}^t=0 \quad \text{or} \quad l_j=0$$

because of the continuity of $h_t{}^t$. If $l_j=0$, then using Theorem A, the conclusion of Theorem A is valid.

In the case which $h_t{}^t=0$, the submanifolds M^{2n} is congruent in E^{2n+1} to a plane E^{2n}, which is naturally imbedded in E^{2n+1}.

In fact, we have from (2.12) $h_{ji}h^{ji}=ph_t{}^t$, which implies $h_{ji}=0$ if $h_t{}^t=0$ and consequently $p=0$ by the definition of p. Thus, the submanifold M^{2n} lies on hypersurface E^{2n+1} of E^{2n+2}.

Taking account of $h_{ji}=0$ we can write (2.17) and (2.24) as

(3.4) $$(1-\lambda^2)k_j{}^tk_{ti}=(\alpha^2+\beta^2)(u_ju_i+v_jv_i),$$

(3.5) $$g^{ji}K_{ji}=-2(\alpha^2+\beta^2),$$

respectively, where $\alpha^2+\beta^2$ is constant because of $g^{ji}K_{ji}=\text{const}$. The tensor $k_j{}^h$ is the second fundamental tensor of M^{2n} immersed in the hypersurface E^{2n+1} with respect to the normal D. We now suppose that $\alpha^2+\beta^2\neq 0$ and restrict ourselves to the open set $M^{2n}{}_o(\subset M^{2n})$ where $1-\lambda^2\neq 0$. Then, taking account of (2.5) and (3.4), we see that $k_j{}^h$ has at each point of $M^{2n}{}_o$ the form

$$(k_i{}^h)=\left(\begin{array}{cc|c} q & 0 & \\ 0 & -q & 0 \\ \hline & 0 & 0 \end{array}\right), \quad q=\sqrt{\alpha^2+\beta^2}$$

with respect to a suitable orthonormal frame. Therefore we can choose in any coordinate neighborhood of $M^{2n}{}_o$, since $(\dim M^{2n})=2n$, a field of frames $\{e_{(1)}, e_{(2)}, \ldots, e_{(2n)}\}$ such that $k_i{}^he^i{}_{(1)}=qe^i{}_{(1)}$, $k_i{}^he^i{}_{(2)}=-qe^i{}_{(2)}$, where $e_{(1)}$ and $e_{(2)}$ are linear combinations of u^h and v^h. On the other hand, we can easily see, by using (1.9) and (1.10) with $h_{ji}=0$, that the distribution spanned in M^{2n}, by u^h and v^h is integrable and totally geodesic in $M^{2n}{}_o$.

Thus, the distribution spanned in $M^{2n}{}_o$ by $e_{(1)}$ and $e_{(2)}$ is also integrable and its integral manifolds are totally geodesic in $M^{2n}{}_o$. Therefore, we can easily verify the fact: the open submanifold $M^{2n}{}_o$ is locally isometric to $S^1(r)\times S^1(r)\times E^{2n-2}$, which is locally flat (cf. [9], [12]). Thus the scalar curvature $g^{ji}K_{ji}$ of M^{2n} vanishes identically in $M^{2n}{}_o$

and hence in M^{2n} because of the continuity of $g^{ji}K_{ji}$. Since $g^{ji}K_{ji}=0$ in M^{2n}, (3.5) implies $\alpha^2+\beta^2=0$, which contradicts the assumption that $\alpha^2+\beta^2\neq 0$. Consequently, we see that $\alpha^2+\beta^2=0$ if $h_t{}^t=0$. Therefore we find, from (3.4), that $k_{ji}=0$ holds identically in M^{2n}. Thus, M^{2n} is totally geodesic in the hyperplane E^{2n+1} and consequently is congruent to a plane $E^{2n}(\subset E^{2n+1}\subset E^{2n+2})$. Hence, using Theorem A we have

THEOREM 3.1. *Let M^{2n} be a complete submanifold of codimension 2 in a $(2n+2)$-dimensional Euclidean space E^{2n+2} such that the scalar curvature of M^{2n} is constant and the mean curvature vector of M^{2n} is parallel in the normal bundle. If $fH=Hf$ and $fK=-Kf$ hold, where H and K are the second fundamental tensors of M^{2n} respectively with respect to C and D, f being the tensor field of type (1, 1) appearing in the induced structure (f, g, u, v, λ) of M^{2n}, then M^{2n} is in E^{2n+2}, provided that $\lambda(1-\lambda^2)$ is non-zero almost everywhere in M^{2n}, congruent to one of the following submanifold:*

$$E^{2n},\quad S^{2n}(r),\quad S^n(r)\times S^n(r),\quad S^l(r)\times E^{2n-l}\quad (l=1, 2\ldots, 2n-1),$$

$$S^k(r)\times S^k(r)\times E^{2n-2k}\quad (k=1, 2, \ldots, n-1),$$

where, $S^k(r)$ denotes a k-dimensional sphere of radius r (>0) imbedded naturally in E^{2n+2}.

THEOREM 3.2. *Let M^{2n} be a complete submanifold of codimension 2 in a $(2n+2)$-dimensional Euclidean space E^{2n+2} such that $fH=Hf$ and there are global unit normals C and D to M^{2n} which are parallel in the normal bundle, where H is the second fundamental tensor of M^{2n} with respect to C, f is the tensor field of type (1, 1) appearing in the induced structure (f, g, u, v, λ) of M^{2n}. If $\nabla_X\lambda=cv(X)$, $\lambda\neq$const. for any vector field X, c being non-zero constant, then the same conclusion as in Theorem 3.1 is valid.*

Proof. Differentiating

$$\nabla_j\lambda=cv_j \tag{3.6}$$

covariantly and taking the skew-symmetric part, we find $\nabla_j v_i-\nabla_i v_j=0$ because c is non-zero constant, from which using (1.10) with $l_j=0$,

$$k_{jt}f_i{}^t-k_{it}f_j{}^t=0.$$

Thus, the assumptions of Proposition 2.1 are constructed and consequently the conclusions of Proposition 2.1 are all valid.

Substituting (2.6), (2.7) and (3.6) into (1.11), we find

$$\alpha=0,\qquad \beta=c+p \tag{3.7}$$

by virtue of the linearly independency of u^i and v^i.

From (2.11) and (2.14) with $l_j=0$, we have respectively

$$(1-\lambda^2)\nabla_j p=(v^t\nabla_t p)v_j,\qquad (1-\lambda^2)\nabla_j p=(u^t\nabla_t p)u_j.$$

By continuity of λ and the fact that $\lambda\neq$ const., we find $\nabla_j p=0$, which implies $p=$const. and consequently $\beta=$const..

Denoting by ρ an eigenvalue of $h_i{}^h$ and by w^h the corresponding eigenvector, we have $h_i{}^t w^i = \rho w^t$, from which, applying $h_t{}^h$ and using (2.12),

$$p h_i{}^h w^i = \rho h_t{}^h w^t, \quad p\rho = \rho^2,$$

that is, $\rho = 0$ or $\rho = p$.

Thus the second fundamental tensor $h_i{}^h$ has only two constant eigenvalues. Let m be the multiplicity of the eigenvalue p, p being assumed to be non-zero, then m is a constant, and we have

$$h_t{}^t = mp, \tag{3.8}$$

which is a constant. But the mean curvature is given by $\frac{1}{2n} h_t{}^t = \frac{1}{2n} mp$ and consequently the mean curvature is a constant too. Since the third fundamental tenor l_j vanishes, (3.2) implies that the mean curvature vector is parallel in the normal bundle.

On the other hand, we see from (2.24) that the scalar curvature $g^{ji}K_{ji}$ of M^{2n} is constant by virtue of β=const., p=const., (3.7) and (3.8). Thus, taking account of Theorem 3.1, we proved Theorem 3.2.

THEOREM 3.3. *Let M^{2n} be a complete submanifold of codimension 2 in a $(2n+2)$-dimensional Euclidean space E^{2n+2} such that $fH = Hf$, $fK = -Kf$ and there are global unit normals C and D to M^{2n} which are parallel in the normal bundle, where H and K are the second fundamental tensors of M^{2n} respectively with respect to C and D, f being the tensor field of type (1, 1) appearing in the induced structure (f, g, u, v, λ) of M^{2n}. If the sectional curvature $K(\gamma)$ with respect to the section spanned by u^h and v^h is constant and $\lambda(1-\lambda^2)$ is non-zero almost everywhere in M^{2n}, then the same conclusion as in Theorem 3.1 is valid.*

Proof. Since C and D are parallel in the normal bundle, the third fundamental tensor l_j vanishes identically. Thus we see, from (2.11) and (2.14), that p is a constant. In the same way as that in the proof of Theorem 3.2, we see that $h_i{}^h$ has only two constant eigenvalues 0, p and consequently

$$h_t{}^t = mp, \tag{3.9}$$

m being the multiplicity of the eigenvalue p.

From (1.12), (2.6) and (2.7) the sectional curvature $K(\gamma)$ with respect to the section γ spanned by u^h and v^h is given by

$$K(\gamma) = -\frac{K_{kjih} u^k v^j u^i v^h}{(u_j u^j)(v_j v^j)} = p^2 - (\alpha^2 + \beta^2),$$

which shows that if $K(\gamma)$ is constant, then

$$p^2 - (\alpha^2 + \beta^2) = \text{const.}. \tag{3.10}$$

Substituting (2.6) and (2.7) into (1.11), we find

$$\nabla_i \lambda = \alpha u_i + (\beta - p) v_i. \tag{3.11}$$

Differentiating (3.11) covariantly, we find

$$\nabla_j\nabla_i\lambda=(\nabla_j\alpha)u_i+\alpha\nabla_j u_i+(\nabla_j\beta)v_i+(\beta-p)\nabla_j v_i,$$

and, hence, using (1.9) and(1.10),

$$\nabla_j\nabla_i\lambda=(\nabla_j\alpha)u_i+\alpha(-h_{jt}f_i{}^t-\lambda k_{ji})+(\nabla_j\beta)v_i+(\beta-p)(-k_{jt}f_i{}^t+\lambda h_{ji}),$$

from which, taking the skew-symmetric part,

$$0=(\nabla_j\alpha)u_i-(\nabla_i\alpha)u_j-2\alpha h_{jt}f_i{}^t+(\nabla_j\beta)v_i-(\nabla_i\beta)v_j. \tag{3.12}$$

Transvecting (3.12) with u^i, we find

$$(1-\lambda^2)(\nabla_j\alpha)=(u^t\nabla_t\alpha)u_j+(u^t\nabla_t\beta-2\alpha\lambda p)v_j. \tag{3.13}$$

Transvecting (3.12) with v^i, we find

$$(1-\lambda^2)(\nabla_j\beta)=(v^t\nabla_t\alpha+2\alpha\lambda p)u_j+(v^t\nabla_t\beta)v_j. \tag{3.14}$$

Multiplying (3.12) by $(1-\lambda^2)$ and substituting (3.13) and (3.14) in the equation obtained, we find

$$2\alpha(1-\lambda^2)h_{jt}f_i{}^t=-(u^t\nabla_t\beta-v^t\nabla_t\alpha-4\alpha\lambda p)(u_jv_i-u_iv_j), \tag{3.15}$$

from which, transvecting with u^i,

$$-2\alpha\lambda p=u^t\nabla_t\beta-v^t\nabla_t\alpha-4\alpha\lambda p,$$

or equivalently

$$u^t\nabla_t\beta-v^t\nabla_t\alpha=2\alpha\lambda p. \tag{3.16}$$

Thus (3.15) becomes

$$\alpha(1-\lambda^2)h_{jt}f_i{}^t=\alpha\lambda p(u_jv_i-u_iv_j),$$

from which, transvecting with $f_h{}^i$,

$$\alpha(1-\lambda^2)h_{jt}(-\delta_h{}^t+u_hu^t+v_hv^t)=-\alpha\lambda^2p(u_ju_h+v_jv_h)$$

or equivalently

$$\alpha(1-\lambda^2)h_{jh}=\alpha p(u_ju_h+v_jv_h). \tag{3.17}$$

Transvecting (3.17) with g^{jh}, we find $\alpha(1-\lambda^2)h_t{}^t=2\alpha p(1-\lambda^2)$, from which, using (3.9),

$$\alpha(m-2)p=0. \tag{3.18}$$

Thus, since m and p are constant, we have

$$(m-2)p=0 \quad \text{or} \quad \alpha=0.$$

We now consider three cases, that is, Case I where $m\neq 2$, $p\neq 0$, Case II where $m=2$ and Case III where $p=0$.

In any case we can prove that the scalar curvature $g^{ji}K_{ji}$ of M^{2n} is constant. In fact, we see from (3.18) that $\alpha=0$ in Case I and consequently β=const. because of (3.10). Thus we see from (2.24) that the scalar curvature $g^{ji}K_{ji}$ of M^{2n} is constant in Case I. In Case II, we have $h_t^t=2p$ because of (3.9) and consequently (2.24) becomes $g^{ji}K_{ji}=2(p^2-\alpha^2-\beta^2)$. Therefore $g^{ji}K_{ji}$=const. because of (3.10). In Case III, taking account of (2.24), (3.9) and (3.10), we see that $g^{ji}K_{ji}=-2(\alpha^2+\beta^2)$=const. Hence, the assumptions of Theorem A are all constructed. This completes the proof of Theorem 3.3.

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