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A FINITARY MATROID WITHOUT A PROPER FINITARY REFINEMENT

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A matroid $\mathfrak{N}$ is a refinement of a matroid $\mathfrak{M}$ if they are defined on a common set E and if the atoms of $\mathfrak{M}$ are also atoms of $\mathfrak{N}$. A matroid is finitary if all its atoms are finite sets.

A construction of a finitary matroid on an infinite set is given which admits of no finitary refinement.

The axiomatic definition of a matroid on an infinite set, subject to a finitary condition, has many forms equivalent to that first given by Rado [2] in terms of a rank function. The one used here is the "circuit" form.

The ordered pair $\mathfrak{M}=(E, \mathfrak{K})$ is a finitary matroid if $\mathfrak{K}$ is a set of incomparable finite non-empty subsets of E satisfying the exchange condition:

(*) For A, $B\in\mathfrak{K}$ with $a\in A-B$ and $b\in A\cap B$ there is $C\in\mathfrak{K}$ with $a\in C\subset A\cup B-\{b\}$.

The elements of $\mathfrak{K}$ are the atoms of the matroid.

$|Z|$ is used to denote the cardinality of the set Z.

For a finite matroid Edmonds [1] has shown that the above exchange axiom (*) may be replaced by the equivalent simpler:

(**) For $A, B\in\mathfrak{K}$ with $b\in A\cap B$ there is $C\in\mathfrak{K}$ with $C\subset A\cup B-b$.

This is a local condition. If Y is a finite subset of E in the finitary matroid $\mathfrak{M}=(E,\mathfrak{K})$, then $\mathfrak{M}_Y=(E\cap Y, \mathfrak{K}\cap P(Y))$ is a finite matroid. ($P(Y)$ is the power set of Y). Then (*) and (**) are equivalent on $\mathfrak{M}_Y$ and hence in $\mathfrak{M}$. Thus Edmonds proof applies to finitary matroid. This greatly simplifies the proofs that follow.

Definition. A sequence $(p_2, p_3, \cdots, p_k)$, of $k-1$ non-negative integers is a *proper partition of the integer* k if,

$$\sum_{j=2}^{k} p_j=k \quad \text{and} \quad \sum_{j=2}^{m} p_j<m \quad \text{for} \quad m<k.$$

Let E be an infinite set and let $(X_i)_{i=1}^{\infty}$ be a partition of E, with each X_i itself an infinite set. For $Z\subset E$ put $Z_s=\bigcup_{j=2}^{s}(Z\cap X_j)$ and put $p_s(A)=|A\cap X_s|$ for each integer $s\geq 2$.

Define the ordered pair $\mathfrak{A}=(E, \mathfrak{K})$ by $A\in\mathfrak{K}$ if and only if:

(i) A is a non-empty finite set, and

(ii) $(p_2(A), \cdots, p_k(A))$ is a proper partition of k, where $k=|A|$.

Lemma. *If $Z\subset E$ and $|Z_s|\geq s$ for some integer s then Z contains a member of $\mathfrak{K}$.*

Proof. Take the first positive integer such that $|Z_s|\geq s$ and take $B\subset Z_s$ with $|B_s|=s$. The sequence $(p_2(B), \cdots, p_s(B))$ is a proper refinement for s:

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$$\sum_{i=2}^{j} p_i(B)=\sum_{i=2}^{j}|B\cap X_i|=|B_j|\leq|Z_j|<j \quad \text{for} \quad j<s$$

by the choice of s, and $\sum_{i=2}^{s} p_i(B)=|B_s|=s$. Thus $B\in\mathfrak{R}$.

THEOREM. *$\mathfrak{A}=(E,\mathfrak{R})$ is a finitary matroid.*

Proof. The elements of $\mathfrak{R}$ are finite by definition, and any two element subset of X_2 is a member of $\mathfrak{R}$.

To show that the elements of $\mathfrak{R}$ are incomparable take $m=|A|<k=|B|$ with $A\subset B\in\mathfrak{R}$. If $A\in\mathfrak{R}$ then $(p_2(A),\cdots,p_m(A))$ is a proper partition of $|A|$. But $m=\sum_{i=1}^{m} p_i(A)\leq\sum_{i=1}^{m} p_i(B)$ with $m<k$, contradicting the condition for $B\in\mathfrak{R}$.

Now take A, B distinct in $\mathfrak{R}$ with $y\in A\cap B$ and put $k=\max(|A|, |B|)$. Then $|A\cup B-y|\geq k$ and by the lemma there is $C\in\mathfrak{R}$ with $C\subset A\cup B-y$. Using Edmond's result referred to above gives the exchange property.

Thus $\mathfrak{A}$ is a finitary matroid.

THEOREM. *Let $\mathfrak{A}=(E,\mathfrak{R})$ be the above finitary matroid. If $\mathfrak{B}$ is any collection of incomparable subsets of the set E and if $\mathfrak{B}$ contains $\mathfrak{R}$ then the only finite subsets in $\mathfrak{B}$ are in $\mathfrak{R}$.*

Proof. Let B be a finite set, $B\in\mathfrak{B}$.

(i) If $|B_s|\geq s$ for some integer s then $B\supset B_s\supset A\in\mathfrak{R}$ for some $A\subset E$ by the lemma, and so $B=A$.

(ii) If $|B_s|<s$ for all integers s, take ν the largest integer such that $B_\nu\neq\phi$. Adjoin $\nu-|B_\nu|$ elements of X_ν to B to obtain the set A. Now $A\cap X_s=B\cap X_s$ for $s<\nu$. Therefore,

$$\sum_{k=2}^{m} p_k(A)=\sum_{k=2}^{m} p_k(B)=|B_m|<m, \quad \text{for} \quad m<\nu,$$

and

$$\sum_{k=2}^{\nu} p_k(A)=\sum_{k=2}^{\nu} p_k(B)+\nu-|B_\nu|=\nu.$$

Thus $A\in\mathfrak{R}\subset\mathfrak{B}$ which contradicts the incomparability of elements of $\mathfrak{B}$.

CONCLUSION. We have shown that the finitary matroid $\mathfrak{A}=(E,\mathfrak{R})$ has no proper refinement. Furthermore, the elements of $\mathfrak{R}$ partition the finite subset of E into two classes: those that contain a member of $\mathfrak{R}$ and those that are contained in a member of $\mathfrak{R}$.

References

[1] Edmonds, J., *Minimum Partition of a Matroid into Independent Subsets*, J. Res. Nat. Bur. Standards, **69B** (1965), 67—72.

[2] Rado, R., *Axiomatic Treatment of Rank in Infinite Sets*, Canad. J. Math. 1 (1949) 337—343.

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