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SUBMANIFOLDS OF CODIMENSION 2 OR 3 WITH PARALLEL SECOND FUNDAMENTAL TENSOR

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§0. Introduction

In a previous paper [4], the present authors have generalized formulas of Nomizu and Smyth [1] to the case of submanifolds of higher codimension and used them to study submanifolds, immersed in a space of constant curvature, whose normal bundle is locally parallelizable and mean curvature vector field is parallel in the normal bundle.

The main purpose of the present paper is to study submanifolds of codimension 2 or 3, immersed in a space of constant curvature, whose second fundamental tensor is parallel with respect to van der Waerden-Bortolotti covariant differentiation. In §1, we recall fundamental formulas of the isometric immersion and, in §2, prove some lemmas concerning submanifolds, immersed in a space of constant curvature, whose second fundamental tensor is parallel and whose holonomy group of the connection induced in the normal bundle is abelian. In §3 and §4, we prove some propositions concerning submanifolds of codimension 2 or 3, immersed in a space of constant curvature, whose second fundamental tensor is parallel.

§1. Submanifolds

Let there be given an n-dimensional differentiable manifold[1] M of class C^∞ covered by a system Σ of coordinate neighborhoods $\{U; y^a\}$[2] and immersed as a submanifold in an m-dimensional Riemannian manifold $\{\tilde{M}, \tilde{g}\}$ with metric tensor $\tilde{g}$. The immersion will be denoted by $i: M \to \tilde{M}$ ($n \geqq 2$, $r = m - n \geqq 1$), that is, M is a submanifold of codimension r in $(\tilde{M}, \tilde{g})$ with immersion $i: M \to \tilde{M}$. The enveloping manifold $\tilde{M}$ is assumed to be covered by a system $\tilde{\Sigma}$ of coordinate neighborhoods $\{\tilde{U}; x^h\}$[3]. Thus we may assume that, for any element U of Σ, $i(U)$ is contained in some element $\tilde{U}$ of $\tilde{\Sigma}$. The immersion $i: M \to \tilde{M}$ is locally expressed by

$$x^h = x^h(y^a) \tag{1.1}$$

in $i(U)(\subset \tilde{U})$, where $\{\tilde{U}; x^h\} \in \tilde{\Sigma}$ and $\{U; y^a\} \in \Sigma$. Differentiating (1.1) partially with respect to y^a, we put

$$B_b{}^h = \partial_b x^h, \qquad \partial_b = \partial/\partial y^b \tag{1.2}$$

in $i(U)$. Then, for each fixed index b, we have along $i(U)$ a local tangent vector field

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1) Manifolds, mappings and geometric objects we discuss are assumed to be differentiable and of class C^∞.

2) The indices a, b, c, d, e run over the range $\{1, 2, \cdots, n\}$ and the summation convention will be used with respect to this system of indices.

3) The indices h, i, j, k run over the range $\{1, 2, \cdots, m\}$ and the summation convention will be used with respect to this system of indices.

B_b to $i(M)$ with components $B_b{}^h$. We consider over $i(U)$ $r(=m-n)$ orthonormal local cross-sections C_y[4] of the normal bundle $N(M)$ of the submanifold M and denote by $C_y{}^h$ components of C_y with respect to $\{\tilde{U};\ x^h\}$, where here and in the sequel we identify $i(M)$ with M and $i(U)$ with U for any $U\in\Sigma$.

The induced metric g of the submanifold M has components $g_{cb}=\tilde{g}_{ji}B_c{}^jB_b{}^i$ in $\{U;\ y^a\}$, where $\tilde{g}_{ji}$ are components of $\tilde{g}$ in $\{\tilde{U};\ x^h\}$. The induced metric $\bar{g}$ of $N(M)$ has components $\bar{g}_{yz}=\tilde{g}_{ji}C_y{}^jC_z{}^i$ in $\{U;\ y^a\}$ with respect to the orthonormal frame $\{C_y\}$, where $\bar{g}_{yz}$ is numerically equal to δ_{yz}. If we put $(B^a{}_i,\ C^x{}_i)=(B_b{}^h,\ C_y{}^h)^{-1}$, then we find $B^a{}_i=B_b{}^hg^{ba}\tilde{g}_{hi}$ and $C^x{}_i=C_y{}^h\bar{g}^{yx}\tilde{g}_{hi}$, where $(g^{cb})=(g_{cb})^{-1}$ and $(\bar{g}^{yx})=(\bar{g}_{yz})^{-1}$.

We denote by $\tilde{\nabla}_cB_b$ and $\tilde{\nabla}_cC_y$ local vector fields, along U, respectively with components

$$\tilde{\nabla}_cB_b{}^h=\partial_cB_b{}^h+\left\{{}_j{}^h{}_i\right\}B_c{}^jB_b{}^i,\qquad \tilde{\nabla}_cC_y{}^h=\partial_cC_y{}^h+\left\{{}_j{}^h{}_i\right\}B_c{}^jC_y{}^i$$

with respect to $\{\tilde{U};\ x^h\}$, where $\left\{{}_j{}^h{}_i\right\}$ are the Christoffel symbols formed with $\tilde{g}_{ji}$. Then the tangential components $(\tilde{\nabla}_cB_b)^T$ of $\tilde{\nabla}_cB_b$ is given by

$$(\tilde{\nabla}_cB_b)^T=\left\{{}_c{}^a{}_b\right\}B_a, \tag{1.3}$$

$\left\{{}_c{}^a{}_b\right\}$ being the Christoffel symbols formed with g_{cb}, and the normal components $(\tilde{\nabla}_cC_y)^\perp$ of $\tilde{\nabla}_cC_y$ by

$$(\tilde{\nabla}_cC_y)^\perp=\Gamma_c{}^x{}_yC_x, \tag{1.4}$$

where $\Gamma_c{}^x{}_y$ denote components of the connection $\bar{\nabla}$ induced in $N(M)$ with respect to the frame $\{C_y\}$ and

$$\Gamma_c{}^x{}_y+\Gamma_c{}^y{}_x=0. \tag{1.5}$$

The normal components $(\tilde{\nabla}_cB_b)^\perp$ of $\tilde{\nabla}_cB_b$ and the tangential component $(\tilde{\nabla}_cC_y)^T$ of $\tilde{\nabla}_cC_y$ have respectively components of the form

$$\nabla_cB_b{}^h=\partial_cB_b{}^h+\left\{{}_j{}^h{}_i\right\}B_c{}^jB_b{}^i-\left\{{}_c{}^a{}_b\right\}B_a{}^h=h_{cb}{}^xC_x{}^h, \tag{1.6}$$

$$\nabla_cC_y{}^h=\partial_cC_y{}^h+\left\{{}_j{}^h{}_i\right\}B_c{}^jC_y{}^i-\Gamma_c{}^x{}_yC_x{}^h=-h_c{}^a{}_yB_a{}^h \tag{1.7}$$

in $\{\tilde{U};\ x^h\}$, where $h_{cb}{}^x$ are components of the *second fundamental tensor* of the submanifold M in $\{U;\ y^a\}$ with respect to the normal frame $\{C_y\}$ and $h_c{}^a{}_y=h_{cb}{}^xg^{ba}\bar{g}_{xy}$. The $h_{cb}{}^x$ is symmetric, i.e., $h_{cb}{}^x=h_{bc}{}^x$.

If we put

$$H=\frac{1}{n}g^{cb}h_{cb}{}^xC_x, \tag{1.8}$$

then we see that H is a global cross-section in $N(M)$, which is called the *mean curvature vector* of the submanifold M. The length $\|H\|$ of H is called the *mean curvature* of the submanifold M. When the mean curvature $\|H\|$ vanishes identically in M, the submanifold M is called a *minimal submanifold.*

Now, we assume that the mean curvature vector H vanishes nowhere in M. Then we can define a unit cross-section $D=H/\|H\|$ in $N(M)$ and hence, putting $D=D^xC_x$, a global tensor field of type $(0,2)$ with components

4) The indices $x,\ y,\ z$ run over the range $\{m+1,\cdots,n\}$ and the summation convention will be used also with respect to this system of indices.

(1.9) $$h_{cb}=h_{cb}{}^{y}D^{x}\bar{g}_{yx}$$

in M. The tensor field h_{cb} is symmetric, i.e., $h_{cb}=h_{bc}$. When the condition

(1.10) $$h_{cb}=\|H\|g_{cb}$$

is satisfied, the submanifold M is said to be *pseudo-umbilical*. When the condition

(1.11) $$h_{cb}{}^{x}=g_{cb}A^{x}$$

is satisfied, $A=A^{x}C_{x}$ being a non-zero cross-section in $N(M)$, the submanifold M is said to be *umbilical*. When $h_{cb}{}^{x}$ vanishes identically in M, the submanifold M is said to be *totally geodesic*.

We denote by $\tilde{K}_{kji}{}^{h}$ components of the curvature tensor of $(\tilde{M},\tilde{g})$ in $\{\tilde{U};\ x^{h}\}$ and by $K_{dcb}{}^{a}$ those of the curvature tensor of (M,g) in $\{U;\ y^{a}\}$. We denote by $\bar{K}_{dcy}{}^{x}$ components of the curvature tensor of the connection $\bar{\nabla}$ induced in $N(M)$ with respect to the normal frame $\{C_{y}\}$, $\bar{K}_{dcy}{}^{x}$ being defined by

(1.12) $$\bar{K}_{dcy}{}^{x}=\partial_{d}\Gamma_{c}{}^{x}{}_{y}-\partial_{c}\Gamma_{d}{}^{x}{}_{y}+\Gamma_{d}{}^{x}{}_{z}\Gamma_{c}{}^{z}{}_{y}-\Gamma_{c}{}^{x}{}_{z}\Gamma_{d}{}^{z}{}_{y}.$$

Then, differentiating (1.6) and (1.7) covariantly and taking account of the Ricci identities, we have the structure equations of the submanifold M immersed in $(\tilde{M},\tilde{g})$ as follows:

(1.13) $$K_{dcb}{}^{a}=\tilde{K}_{dcb}{}^{a}+(h_{d}{}^{a}{}_{x}h_{cb}{}^{x}-h_{c}{}^{a}{}_{x}h_{db}{}^{x}),$$

(1.14) $$0=\tilde{K}_{dcb}{}^{x}+\nabla_{d}h_{cb}{}^{x}-\nabla_{c}h_{db}{}^{x},$$

(1.15) $$\bar{K}_{dcy}{}^{x}=\tilde{K}_{dcy}{}^{x}+h_{de}{}^{x}h_{c}{}^{e}{}_{y}-h_{ce}{}^{x}h_{d}{}^{e}{}_{y},$$

where we have put

$$\tilde{K}_{dcb}{}^{a}=\tilde{K}_{kji}{}^{h}B_{d}{}^{k}B_{c}{}^{j}B_{b}{}^{i}B^{a}{}_{h},\qquad \tilde{K}_{dcb}{}^{x}=\tilde{K}_{kji}{}^{h}B_{d}{}^{k}B_{c}{}^{j}B_{b}{}^{i}C^{x}{}_{h},$$

$$\tilde{K}_{dcy}{}^{x}=\tilde{K}_{kji}{}^{h}B_{d}{}^{k}B_{c}{}^{j}C_{y}{}^{i}C^{x}{}_{h}$$

and

(1.16) $$\nabla_{d}h_{cb}{}^{x}=\partial_{d}h_{cb}{}^{x}-\left\{{}_{d}{}^{a}{}_{c}\right\}h_{ab}{}^{x}-\left\{{}_{d}{}^{a}{}_{b}\right\}h_{ca}{}^{x}+\Gamma_{d}{}^{x}{}_{y}h_{cb}{}^{y}.$$

If the enveloping manifold $(\tilde{M},\tilde{g})$ is of constant curvature c, i.e., if

$$\tilde{K}_{kji}{}^{h}=c(\delta_{k}^{h}g_{ji}-\delta_{j}^{h}g_{ki}),$$

then we have, substituting the equation above in (1.13), (1.14) and (1.15) respectively,

(1.17) $$K_{dcb}{}^{a}=c(\delta_{d}^{a}g_{cb}-\delta_{c}^{a}g_{db})+(h_{d}{}^{a}{}_{x}h_{cb}{}^{x}-h_{c}{}^{a}{}_{x}h_{db}{}^{x}),$$

(1.18) $$0=\nabla_{d}h_{cb}{}^{x}-\nabla_{c}h_{db}{}^{x},$$

(1.19) $$\bar{K}_{dcy}{}^{x}=h_{de}{}^{x}h_{c}{}^{e}{}_{y}-h_{ce}{}^{x}h_{d}{}^{e}{}_{y}.$$

Transvecting (1.17) with g^{cb} and contracting the indices a and d, we find

$$K=n(n-1)c+n^{2}\|H\|^{2}-h_{cb}{}^{x}h^{cb}{}_{x},$$

where $K=g^{cb}K_{acb}{}^{a}$ is the scalar curvature of (M,g) and $h^{cb}{}_{x}=h_{da}{}^{y}g^{dc}g^{ab}\bar{g}_{xy}$. Thus, using the equation above, we have (cf. [3])

LEMMA 1.1. *Let M be a submanifold immersed in a Riemannian manifold of constant* *curvature c. Then, if the scalar curvature K of (M, g) with induced metric g satisfies* $K \geqq n(n-1)c$, *(dim M=n), and if M is a minimal submanifold, then M is totally geodesic.*

A Riemannian manifold (M, g) is said to be *irreducible*, if its linear holonomy group is irreducible as a linear group. A Riemannian manifold is said to be *locally irreducible* if any open submanifold of (M, g) is irreducible as a Riemannian manifold having the restriction of g as metric tensor. A submanifold M is said to be *(locally) irreducible*, if the Riemannian manifoid (M, g) with induced metric g is (locally) irreducible.

If the second fundamental tensor $h_{cb}{}^x$ satisfies the condition

$$\nabla_d h_{cb}{}^x = 0, \tag{1.20}$$

then we say that the submanifold M has *parallel second fundamental tensor*. The Ricci identity and (1.20) imply

$$-K_{edc}{}^a h_{ab}{}^x - K_{edb}{}^a h_{ca}{}^x + \bar{K}_{edy}{}^x h_{cb}{}^y = 0. \tag{1.21}$$

Thus, if $\nabla_d h_{cb}{}^x = 0$, then (1.21) holds.

We assume that the submanifold M has parallel second fundamental tensor. Then, using (1.5), (1.8), (1.9) and (1.20), we find

$$\nabla_d h_{cb} = \partial_d h_{cb} - \left\{ {}_d{}^a{}_c \right\} h_{ab} - \left\{ {}_d{}^a{}_b \right\} h_{ca} = 0.$$

Thus, if the submanifold M is, moreover, irreducible, then we find $h_{cb} = a g_{cb}$, a being a constant, because of $h_{cb} = h_{bc}$. Consequently, we have

PROPOSITION 1.1. *Let M be a submanifold with parallel second fundamental tensor. If M is irreducible and not a minimal submanifold, then it is a pseudo-umbilical submanifold with constant mean curvature.*

We define the covariant derivative of the curvature tensor of the connection $\bar{\nabla}$ induced in $N(M)$ as a tensor field with components

$$\begin{aligned} \nabla_e \bar{K}_{dcy}{}^x = \partial_e \bar{K}_{dcy}{}^x - \left\{ {}_e{}^a{}_d \right\} \bar{K}_{acy}{}^x - \left\{ {}_e{}^a{}_c \right\} \bar{K}_{day}{}^x \\ - \Gamma_e{}^z{}_y \bar{K}_{dcz}{}^x + \Gamma_e{}^x{}_z \bar{K}_{dcy}{}^z. \end{aligned} \tag{1.22}$$

Then, differentiating (1.17) and (1.19) covariantly and taking account of (1.22), we have

LEMMA 1.2. *Let M be a submanifold immersed in a Riemannian manifold of constant curvature and assume that M has parallel second fundamental tensor. Then*

$$\nabla_e \bar{K}_{dcy}{}^x = 0 \tag{1.23}$$

holds and the Riemannian manifold (M, g) with the induced metric g is locally symmetric, i.e., $\nabla_e K_{dcb}{}^a = 0$, *the scalar curvature K of (M, g) being constant.*

If a submanifold M immersed in a Riemannian manifold of constant curvature has parallel second fundamental tensor, then, by Lemma 1.2, (M, g) is locally symmetric, g being the induced metric of M. Thus, if such a submanifold M is irreducible, then M is locally irreducible.

§ 2. Normal bundles with abelian holonomy group

Let there be given a submanifold M of codimension r of $(\tilde{M}, \tilde{g})$. We assume that the holonomy group Φ of the connection $\bar{\nabla}$ induced in the normal bundle $N(M)$ is abelian. Then Φ is similar to a subgroup of the linear group Ψ of all matrices of the form

$$\begin{pmatrix} A_1 & & & 0 \\ & \ddots & & \\ & & A_s & \\ 0 & & & A_0 \end{pmatrix}; \qquad A_1, \cdots, A_s \in R(2), \quad 0 \leqq 2s \leqq r, \tag{2.1}$$

in the orthogonal group $O(r)$, where $R(2)$ denotes the group of rotations and A_0 a diagonal $(r-s,\ r-s)$-matrix with diagonal elements $+1$ or -1. Thus the Lie algebra of the holonomy group Φ is a subalgebra of the Lie algebra of all matrices of the form

$$\begin{pmatrix} B_1 & & & 0 \\ & \ddots & & \\ & & B_s & \\ 0 & & & O_{r-s} \end{pmatrix}; \qquad B_\alpha = \begin{pmatrix} 0 & -\theta_\alpha \\ \theta_\alpha & 0 \end{pmatrix}, \tag{2.2}$$

$$\alpha = 1, \cdots, s,$$

where O_{r-s} denotes the zero $(r-s,\ r-s)$-matrix. Therefore there is, along each coordinate neighborhood U of M, orthonormal cross-sections C_y of $N(M)$ satisfying the following conditions (A) and (B):

(A) The connection $\bar{\nabla}$ induced in $N(M)$ has in U components $\Gamma_c{}^x{}_y$ of the form

$$(v^d \Gamma_d{}^x{}_y) = \begin{pmatrix} L_1 & & & 0 \\ & \ddots & & \\ & & L_s & \\ 0 & & & O_{r-s} \end{pmatrix} \tag{2.3}$$

with respect to the frame $\{C_y\}$, v^a being an arbitrary vector field in M, where

$$L_\alpha = \begin{pmatrix} 0 & -v^c l_{(\alpha)c} \\ v^c l_{(\alpha)c} & 0 \end{pmatrix}, \quad \alpha = 1, \cdots, s, \tag{2.4}$$

$l_{(\alpha)c}$ being, for each fixed index α, a local covector field in U, which is non-zero;

(B) For any intersecting coordinate neighborhoods U and U' of M, let $\{C_y\}$ and $\{C'_y\}$ be the frame of orthonormal cross-sections of $N(M)$ along U and U' respectively. Then the matrix $\varphi = (\varphi_y{}^x)$ belongs to Φ, where $C'_y = \varphi_y{}^x C_x$ in $U \cap U'$.

Such an orthonormal frame $\{C_y\}$ as constructed above is called an ***adapted normal frame*** in U.

Taking account of (1.12) and using (2.3) and (2.4), we see that the curvature tensor of the connection $\bar{\nabla}$ has in U components $\bar{K}_{dcy}{}^x$ of the form

$$(v^d u^c \bar{K}_{dcy}{}^x) = \begin{pmatrix} \Omega_1 & & & 0 \\ & \ddots & & \\ & & \Omega_s & \\ 0 & & & O_{r-s} \end{pmatrix} \tag{2.5}$$

with respect to an adapted normal frame $\{C_y\}$, u^a and v^a being arbitrary vector fields in M, where

$$\Omega_\alpha=\begin{pmatrix} 0 & -v^d u^c(\partial_d l_{(\alpha)c}-\partial_c l_{(\alpha)d}) \\ v^d u^c(\partial_d l_{(\alpha)c}-\partial_c l_{(\alpha)d}) & 0 \end{pmatrix}, \tag{2.6}$$

$\alpha=1, \cdots, s$ and

$$F_{(\alpha)}=(\partial_d l_{(\alpha)c}-\partial_c l_{(\alpha)d})dy^d\wedge dy^c \tag{2.7}$$

is a 2-form defined globally in M, for each fixed index α, since the matrix (2.5) with Ω_α defined by (2.6) is invariant under the group Ψ of the form (2.1). We may assume that each of $F_{(\alpha)}$ is a non-zero 2-form. If $\dim \Phi=d>0$, then there are d members of $F_{(\alpha)}$'s which are linearly independent with respect to constant coefficients.

We assume, moreover, that the submanifold M has parallel second fundamental tensor and that the enveloping manifold is of constant curvature. Then, by Lemma 1.2, we find $\nabla_e \bar{K}_{dcy}{}^x=0$, from which,

$$\nabla_d F_{(\alpha)cb}=0, \qquad \alpha=1, \cdots, s. \tag{2.8}$$

Thus each of $F_{(\alpha)}$ is in M a harmonic tensor of degree 2. Hence we have

LEMMA 2.1. *Let M be a submanifold satisfying the conditions:*

(P) *M is immersed in a Riemannian manifold of constant curvature c;*

(Q) *M has parallel second fundamental tensor;*

(R) *The holonomy group Φ of the connection $\bar{\nabla}$ induced in the normal bundle $N(M)$ of M is abelian.*

If M is compact, and, if $B_2=0, B_2$ being the second Betti number of M, then $\dim \Phi=0$.

We suppose that a submanifold M satisfying the conditions (P), (Q) and (R) is irreducible. Then, taking account of (2.8), we see that, if $\dim \Phi>0$, there is at least one member, say F, of $F_{(\alpha)}$'s such that F is a non-zero 2-form and kF, k being a non-zero constant, is the fundamental 2-form J of a certain Kaehlerian structure (g, J) in M, where g is the induced metric of M. Thus we have

LEMMA 2.2. *If a submanifold M satisfying tha conditions (P), (Q) and (R) is irreducible, and if $\dim \Phi>0$, then there is in M a certain Kaehlerian sturucture (g, J), where g is the induced metric of M.*

Next, we assume that a submanifold M satisfying the conditions (P), (Q) and (R) is irreducible and that M admits no Kaehlerian structure (g, J), g being the induced metric of M. Then, by Lemma 2.2, we find $\dim \Phi=0$, that is, the curvature tensor $\bar{K}_{dcy}{}^x$ of the connection $\bar{\nabla}$ vanishes. Thus, for each point of M, there is a coordinate neighborhood U containing this point such that there is in U an adapted normal frame $\{C_y\}$, with respect to which $\bar{\nabla}$ has vanishing components, i.e., $\Gamma_d{}^x{}_y=0$. Thus each of the second fundamental tensor $h_{cb}{}^x$ is, for each fixed index x, a symmetric tensor field of type (0,2) in U and has vanishing covariant derivative, i.e.,

$$\nabla_d h_{cb}{}^x=\partial_d h_{cb}{}^x-\left\{{}_d{}^a{}_c\right\}h_{ab}{}^x-\left\{{}_d{}^a{}_b\right\}h_{ca}{}^x=0$$

because of (1.16) with $\Gamma_d{}^x{}_y=0$. On the other hand, the Riemannian manifold (U, g), which is an open submanifold of (M, g), is irreducible, because (M, g) is irreducible and symmetric. Thus we obtain

$$h_{cb}{}^{x}=g_{cb}A^{x},$$

where $A=A^{x}C_{x}$ is a parallel local cross-section in $N(M)$. Summing up, we have

LEMMA 2.3. *If a submanifold M satisfying the conditions (P), (Q) and (R) is irreducible, and if M admits no Kaehlerian structure (g, J), g being the induced metric of M, then M is umbilical or totally geodesic.*

As a corollary to Lemma 2.3, we have

LEMMA 2.4. *If a submanifold M satisfying the conditions (P), (Q) and (R) is odd-dimensional and irreducible, then M is umbilical or totally geodesic.*

Let M be differentiably homeomorphic to a natural sphere. Then M admits no Kaehlerian structure, if $\dim M>2$. On the other hand, any Riemannian manifold having a natural sphere as its underlying manifold is irreducible. Thus, from Lemma 2.3, we have

LEMMA 2.5. *If a submanifold M satisfying the conditions (P), (Q) and (R) is differentiably homeomorphic to a natural sphere, and if* $\dim M>2$, *then M is umbilical or totally geodesic.*

From now on, in this section, we denote by M a 2-dimensional submanifold satisfying the conditions (P), (Q) and (R). In a coordinate neighborhood U of M, we take an adapted normal frame $\{C_{y}\}$, which satisfies the conditions (A) and (B). We suppose that the 2-form

$$\theta=(\partial_{d}l_{(1)c}-\partial_{c}l_{(1)d})dy^{d}\wedge dy^{c}$$

corresponding to Ω_{1} and appearing in (2.5) is not zero. Since $\dim M=2$, we can put in U

$$\theta=-f dy^{1}\wedge dy^{2}, \tag{2.9}$$

the function f being zero nowhere in U because of $\nabla_{e}\bar{K}_{dcy}{}^{x}=0$ given in Lemma 1.2. On the other hand, since $\dim M=2$, we can put in U

$$K_{dcb}{}^{a}=A(\delta_{d}^{a}g_{cb}-\delta_{c}^{a}g_{db}), \tag{2.10}$$

A being a function in M. Since the submanifold M satisfies the condition (Q), we have $\nabla_{d}h_{cb}{}^{x}=0$, which implies (1.21). The condition (1.21) reduces, for $e=1$, $d=2$, $x=m+1$, to

$$-A(P_{1b}g_{2c}-P_{2b}g_{1c}+P_{1c}g_{2b}-P_{2c}g_{1b})-fQ_{cb}=0 \tag{2.11}$$

and, for $e=1$, $d=2$, $x=m+2$, to

$$-A(Q_{1b}g_{2c}-Q_{2b}g_{1c}+Q_{1c}g_{2b}-Q_{2c}g_{1b})+fP_{cb}=0 \tag{2.12}$$

by means of (2.9) and (2.10), where we have put $P_{cb}=h_{cb}{}^{m+1}$ and $Q_{cb}=h_{cb}{}^{m+2}$.

Now, if we take account of the fact that the mean curvature vector $H=(1/n)g^{cb}h_{cb}{}^{x}C_{x}$ is parallel in $N(M)$, we have $g^{cb}h_{cb}{}^{m+1}=0$ and $g^{cb}h_{cb}{}^{m+2}=0$ in U. Thus, putting $P_{cb}=h_{cb}{}^{m+1}$ and $Q_{cb}=h_{cb}{}^{m+2}$, we obtain $g^{cb}P_{cb}=0$ and $g^{cb}Q_{cb}=0$. Thus, taking suitably in U

an orthonormal frame $\{e_1, e_2\}$ tangent to M, we have

$$(P_{cb})=\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \qquad (Q_{cb})=\begin{pmatrix} c & b \\ b & -c \end{pmatrix} \tag{2.13}$$

with respect to $\{e_1, e_2\}$. Substituting (2.13) into the tensor equations (2.11) and (2.12) with respect to $\{e_1, e_2\}$, i.e., the tensor equation (2.11) and (2.12) with $g_{cb}=\delta_{cb}$, we find

$$2aA+fc=0, \quad fb=0 \quad \text{and} \quad 2Ab+fa=0, \quad 2Ac=0,$$

respectively. From these equations, we have $a=b=c=0$, because f is zero nowhere in U. Therefore $P_{cb}=Q_{cb}=0$ in U, that is, $h_{cb}{}^{m+1}=h_{cb}{}^{m+2}=0$. If we substitute $h_{cb}{}^{m+1}=h_{cb}{}^{m+2}=0$ into (1.19), then in U we obtain $\bar{K}_{dc1}{}^{2}=0$, i.e., $\Omega_1=0$, which contradicts the assumption that the 2-form Ω_1 is not zero in U. Consequently, the 2-form Ω_1 necessarily vanishes in M. Similarly, all other 2-forms Ω_a's appearing in (2.5) vanish in M. That is to say, the curvature tensor $\bar{K}_{dcy}{}^{x}$ of the connection $\bar{\nabla}$ induced in $N(M)$ vanishes identically in M. Summing up, we have

LEMMA 2.6. *Let M be a 2-dimensional submanifold satisfying the conditions (P), (Q) and (R). Then the curvature tensor of the connection $\bar{\nabla}$ induced in $N(M)$ is zero in M. If M is irreducible (not flat with respect to the induced metric g), then M is umbilical or totally geodesic.*

If, in Lemma 2.6, the enveloping manifold is a Euclidean space or a sphere, and if the submanifold M is complete, connected and irreducible, then M is a 2-dimensional natural sphere. If, in Lemma 2.6, the enveloping manifold is a Euclidean space or a sphere, and if the submanifold M is complete, connected and flat with respect to the induced metric g, then M is a a plane or a pythagorean product of the form $S^1(a)\times S^1(b)$, where $S^1(a)$ denotes a natural circle of radius a.

§3. Submanifolds of codimension 2

We need, in the sequel, the following theorems A and B proved in [2] (cf. [4]):

THEOREM A. *Let M be a connected and complete submanifold of dimension n with parallel second fundamental tensor immersed in a Euclidean space R^m of dimension m $(1<n<m)$ and suppose that the curvature tensor of the connection induced in the normal bundle $N(M)$ of M is zero. Then M is an n-dimensional sphere $S^n(a)$, n-dimensional plane R^n, a pythagorean products of the form*

$$S^{p_1}(a_1)\times\cdots\times S^{p_N}(a_N);$$

$$p_1+\cdots+p_N=n; \quad p_1\geqq\cdots\geqq p_N\geqq 1; \quad 1<N\leqq m-n$$

or a pythagorean product of the form

$$S^{p_1}(a_1)\times\cdots\times S^{p_N}(a_N)\times R^p;$$

$$p_1+\cdots+p_N+p=n; \quad p>0; \quad p_1\geqq\cdots\geqq p_N\geqq 1; \quad 1<N\leqq m-n,$$

where $S^p(a)$ $(p>1)$ denotes a p-dimensional sphere with radius a naturally embedded.

$S^1(a)$ a natural circle of radius a naturally embedded and R^p a p-dimensional plane.

THEOREM B. *Let M be a connected and complete submanifold of dimension n with parallel second fundamental tensor immersed in an m-dimensional sphere $S^m(a)$ with radius a $(0<a,\ 1<n<m)$ and suppose that the curvature tensor of the connection induced in the normal bundle N(M) of M is zero. Then M is a sphere $S^n(b)$ $(0<b\leqq a)$, a pythagorean product of the form*

$$S^{p_1}(a_1)\times\cdots\times S^{p_N}(a_N),\qquad a_1^2+\cdots+a_N^2=a^2:$$

$$p_1+\cdots+p_N=n;\quad p_1\geqq\cdots\geqq p_N\geqq 1;\quad N=m-n+1$$

or a pythagorean product of the form

$$S^{p_1}(a_1)\times\cdots\times S^{p_N}(a_N),\qquad a_1^2+\cdots+a_N^2\leqq a^2;$$

$$p_1+\cdots+p_N=n;\quad p_1\geqq\cdots\geqq p_N\geqq 1;\quad 1<N\leqq m-n.$$

Let M be a submanifold of codimension 2 in a Riemannian manifold. Then the normal bundle $N(M)$ is of dimension 2. Thus the holonomy group Φ of the connection induced in $N(M)$ is a subgroup of $O(2)$ and hence abelian. Thus, from Lemma 2.1, Theorems A and B, we have

PROPOSITION 3.1. *Let M be a submanifold of codimension 2 immersed in a Euclidean space and assume that the second fundamental tensor is parallel. If M is compact and has vanishing second Betti number B_2, then M is a sphere $S^n(a)$ $(n\neq 2)$ or a pythagorean product of the form*

$$S^{p_1}(a_1)\times S^{p_2}(a_2);$$

$$p_1+p_2=n;\quad p_1\geqq 3,\ p_2>1,\ p_2\neq 2,$$

where $\dim M=n$, provided that M is connected.

PROPOSITION 3.2. *If, in Proposition 3.1, the enveloping manifold is a sphere $S^{n+2}(a)$, then M is a sphere $S^n(b)$ $(n\neq 2,\ 0<b\leqq a)$, a pythagorean product of the form*

$$S^{p_1}(a_1)\times S^{p_2}(a_2)\times S^{p_3}(a_3),\qquad a_1^2+a_2^2+a_3^2=a^2;$$

$$p_1+p_2+p_3=n;\qquad p_1\geqq p_2\geqq 3,\ p_3\geqq 1,\ p_3\neq 2$$

or a pytagorean product of the form

$$S^{p_1}(a_1)\times S^{p_2}(a_2),\qquad a_1^2+a_2^2\leqq a^2;$$

$$p_1+p_2=n;\qquad p_1\geqq 3,\ p_2\geqq 1,\ p_2\neq 2,$$

where $\dim M=n$, provided that M is connected.

From Lemma 2.4, we have

PROPOSITION 3.3. *Let M be an odd-dimensional submanifold of codimension 2 satisfying the conditions (P) and (Q). If M is irreducible, then M is umbilical or totally geodesic.*

From Lemmas 2.5 and 2.6, we have

PROPOSITION 3.4. *Let M be a submanifold of codimension 2 satisfying the conditions*

(P) and (Q). If M is differentiably homeomorphic to a natural sphere, then M is umbilical or totally geodesic.

If, in Propositions 3.3 and 3.4, the enveloping manifold is a Euclidean space or a sphere, and if M is connected and complete, then the submanifold is a sphere naturally embedded.

If a submanifold M of codimension 2 satisfying the conditions (P) and (Q) is not minimal, then the mean curvature vector H is non-vanishing and parallel in $N(M)$. Thus the holonomy group Φ of the connection $\bar{\nabla}$ induced in $N(M)$ is of dimension 0, that is, the curvature tensor $\bar{K}_{dcy}{}^{x}$ of $\bar{\nabla}$ vanishes identically. Therefore, for such a submanifold M, we have the same results as those stated in Theormes A and B with $m=n+2$, where $\dim M=n$.

Taking account of Lemma 1.1, we have, for a submanifold of codimension 2 satisfying the conditions (P) and (Q), the same results as stated in Theorems A and B with $m=n+2$, where $\dim M=n$, if the scalar curvature K of M satisfies $K\geqq n(n-1)c$, c being the curvature of the enveloping manifold.

§4. Submanifolds of codimension 3.

Let there be given a submanifold M of codimension 3 satisfying the conditions (P) and (Q). If we assume that M is not a minimal submanifold, then we see that the mean curvature vector H is non-vanishing and parallel in $N(M)$. Thus, since $N(M)$ is 3-dimensional, the holonomy group Φ of the connection induced in $N(M)$ is a subgroup of $O(2)$ and hence abelian. Thus, taking account of Lemmas 1.1, 2.1, Theorems A and B, we have

Proposition 4.1. *Let M be a submanifold of codimension* 3 *immersed in a Euclidean space and assume that M has parallel second fundamental tensor. Assume that M is not a minimal submanifold or that the scalar curvature K of M is non-negative. If M is compact and has vanishing second Betti number* B_2, *then M is a sphere* $S^n(a)$ $(n\neq 2)$, *a pythagorean product of the form*

$$S^{p_1}(a_1)\times S^{p_2}(a_2)\times S^{p_3}(a_3);$$

$$p_1+p_2+p_3=n;\qquad p_1\geqq p_2\geqq 3,\ p_1\geqq 1,\ p_1\neq 2$$

or a pythagorean product of the form

$$S^{p_1}(a_1)\times S^{p_2}(a_2);$$

$$p_1+p_2=n;\quad p_1\geqq 3,\ p_2\geqq 1,\ p_2\neq 2,$$

where $\dim M=n$, *provided that M is connected.*

Proposition 4.2. *If, in Proposition* 4.1, *the enveloping manifold is a sphere* $S^{n+3}(a)$, *then M is a sphere* $S^n(b)$ $(n\neq 2,\ 0<b\leqq a)$, *a pythagorean product of the form*

$$S^{p_1}(a_1)\times S^{p_2}(a_2)\times S^{p_3}(a_3)\times S^{p_4}(a_4),\qquad a_1^2+a_2^2+a_3^2+a_4^2=a^2;$$

$$p_1+p_2+p_3+p_4=n;\quad p_1\geqq p_2\geqq p_3\geqq 3,\ p_4\geqq 1,\ p_4\neq 2,$$

a pythagorean product of the form

$$S^{p_1}(a_1)\times S^{p_2}(a_2)\times S^{p_3}(a_3),\quad a_1{}^2+a_2{}^2+a_3{}^2\leqq a^2;$$

$$p_1+p_2+p_3=n;\qquad p_1\geqq p_2\geqq 3,\ p_3\geqq 1,\ p_3\neq 2$$

or a pythagorean product of the form

$$S^{p_1}(a_1)\times S^{p_2}(a_2),\qquad a_1{}^2+a_2{}^2\leqq a^2;$$

$$p_1+p_2=n;\qquad p_1\geqq 3,\ p_2\geqq 1,\ p_2\neq 2,$$

where $\dim M=n$, *provided that* M *is connected.*

We have the following Propositions 4.3 and 4.4 by the same devices as developed in the proofs of Propositions 3.3 and 3.4 respectively.

PROPOSITION 4.3. *Let* M *be an odd-dimensional submanifold of condimension* 3 *satisfying the conditions* (P) *and* (Q) *and assume that* M *is not minimal submanifold. If* M *is irreducible, then* M *is umbilical or totally geodesic.*

PROPOSITION 4.4. *Let* M *be a submanifold of codimension* 3 *satisfying the conditions* (P) *and* (Q) *and assume that* M *is not a minimal submanifold. If* M *is differentiably homeomorphic to a natural sphere, then* M *is umbilical or totally geodesic.*

If we take account of Lemma 1.1, we see that Propositions 4.3 and 4.4 are established, replacing the assumption that M is not a minimal submanifold by another assumption that the scalar curvature K of M satisfies $K\geqq n(n-1)c$, $\dim M=n$, c being the curvature of the enveloping manifold.

If, in Propositions 4.3 and 4.4, the enveloping manifold is a Euclidean space or a sphere, and if the submanifold M is connected and complete, then M is a sphere naturally embedded.

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