A Note on H-closed Spaces

Kim Heung Kee

§1. Introduction

Since Alexandroff and Urysohn [4] introduced H-closed spaces by categorical means, many other characterizations for H-closed spaced were given. In this paper we characterize it by means of regular open filters and investigate the properties of H-closed spaces by filter concepts.

A Hausdorff space X is called H-closed provided that for every open covering of X, there is a finite subfamily the union of closures of members of which covers X. a subset U[F] of a topological space is said to be regular open (regular closed) (for brevity we write r.o.) if $U^{-0}=U$ (this means Int *cl* U=U) [$F^{0-}=F$]. It is to be understood that the interior of a closed set, complement of a regular closed set, intersection of finite regular open sets are also r.o. ([3]p. 92). We denote the set of all r.o. subsets of X by R ($\mathfrak{X}$). Spaces mean only Hausdorff spaces.

§2. Regular open filters.

1. Defintion a nonempty subfamily $\mathfrak{F}$ of R ($\mathfrak{X}$) is called r.o. filter provided that

(i) $\phi \notin \mathfrak{F}$;

(ii) if U_1, $U_2 \in \mathfrak{F}$, then $U_1 \cap U_2 \in \mathfrak{F}$: and

(iii) if $U \in \mathfrak{F}$, $V \in R(\mathfrak{X})$, and $U \subset V$, then $V \in \mathfrak{F}$

As usual, maximal r.o. filter means the maximal element among the r.o. filters. We can characterize it as follows.

2. Lemma A r.o. filter $\mathfrak{F}$ is maximal iff for every $A \in \mathfrak{B}(\mathfrak{X})$ such that $A \cap U \neq \phi$ for every $U \in \mathfrak{F}$, $A \in \mathfrak{F}$.

proof. Let $\mathfrak{n}$ be a r.o. filter strictly finer than F, say $A \in \mathfrak{n} - \mathfrak{F}$. Then A meets every member of $\mathfrak{F}$ but $A \notin \mathfrak{F}$.

Conversely, assume $A \notin \mathfrak{F}$. Then $\mathfrak{F} \cup \{A\}$ has the FIP (finite intersection property), hence there is a r.o. filter containing $\mathfrak{F}$ properly.

§3. H-closed spaces.

3. Theorem The following are equivalent for Hausdorff spaces:

(1) X is H-closed.

(2) For every Hausdorff space Y, and each continuous function $f: X \to Y$, $f(X)$ is closed in Y (i.e., X is absolutely closed.)

(3) For every family $\mathfrak{F} = \{F_\alpha | \alpha \in \mathfrak{A}\}$ of closed subsets of X with $\bigcap_{\alpha \in \mathfrak{A}} F_\alpha = \phi$, there is a finite subfamily $\{F_{\alpha_1}, \cdots, F_{\alpha_n}\}$ of F with $\bigcap F^0_{\alpha_i} = \phi$.

(3)′ Every family $\mathfrak{F}$ of closed subsets of X such that $\mathfrak{F}^\circ$ (the set of interiors of members of $\mathfrak{F}$) has Finte Intersection Property is fixed.

(4) Each r.o. filterbase on X has at least one accumulation point.

(5) Each maximal r.o. filterbase on X has an accumulation point (neccessarily unique).

(6) The space X* with minimal r.o.-equivalent topology is H-closed.

(7) For each r.o. open covering $\mathfrak{U}$ of X, there is a finite subcollection $\mathfrak{A}$ such that $\bigcup \overline{\mathfrak{A}} = X$.

proof.

(1⇒2) clear.

(2⇒1) [1]

(1↔3) clear.

(3⇒4) Let $\mathfrak{U} = \{U_\alpha | \alpha \in A\}$ be a r.o. filter base in X. $\mathfrak{U}$ is a family of closed sets such that $\overline{\mathfrak{U}}^\circ = \mathfrak{U}$ has the FIP. By (3), $\overline{\mathfrak{U}}$ has nonempty *intersection i.e.*, $\overline{\mathfrak{U}}$ *has an accumulation point.*

(4⇒3) Suppose $\mathfrak{F}$ is a family of closed sets with

$\mathfrak{F}^\circ$ FIP. Then $\mathfrak{F}^\circ$ is a r.o. filter base (since $A^\circ \cap B^0=(A\cap B)^0$), by (4), $\cap\mathfrak{F}^{\circ -}\neq\phi$. Thus $\cap\mathfrak{F}\supset\cap\mathfrak{F}^{\circ -}\neq\phi$.

(4⇒5) clear.

(5⇒4) Let $\mathfrak{U}$ be a r.o. filter base on X. Let **T**=the set of all filter base on X finer than $\mathfrak{U}$. To show **T** is inductive:

Let **S** be a chain in **T** and $\mathfrak{W}=\bigcup_{\mathfrak{V}\in S}\mathfrak{A}$. Then

① $\phi\notin\mathfrak{W}$

② $A_\alpha, A_\beta\in\mathfrak{W}\Rightarrow{}^\exists\mathfrak{A}\in\mathbf{S}$: $A_\alpha, A_\beta\in\mathfrak{A}$ ${}^\exists A_\gamma\in\mathfrak{A}\subset\mathfrak{W}$ s.t. $A_\gamma\subset A_\alpha\cap A_\beta$. This shows that **T** is inductive. By Zorn's lemma, there is a maximal element $\mathfrak{M}$ in **T**.

(1↔6) Clear from(1↔7)

(1↔7) ([2] p.10)

§4. Properties of H-closed spaces.

Although H-closed space is defined very similarly to compact spaces, here is a crucial distinction between them:

4. Theorem A H-closed space is regular iff it is compact.

proof. For non-trivial part, let $\mathfrak{U}$ be an open covering of a H-closed space X. By the regularity of X, there is an open cover $\mathfrak{A}$ such that $\overline{\mathfrak{A}}$ refines $\mathfrak{U}$. Let $\{V_i\}\ 1\leqq i\leqq n\}$ be a subfamily of $\mathfrak{A}$ such that $\bigcup\overline{V}_i=X$. Then $\{U_i\in\mathfrak{U}\mid\overline{V}_i\subset U_i,\ 1\leqq i\leqq n\}$ is the desired finite subcovering of $\mathfrak{U}$.

5. Theorem Continuous image of a H-closed subset is H-closed.

proof. Let X be H-closed, and $f: X\to Y$ continuous. Let $\{U_\alpha\}$ be any open covering of $f(X)$; then $\{f^{-1}(U_\alpha)\}$ is an open covering of X, and so can be reduced to a finite closure covering, $\{\overline{f^{-1}(U_{\alpha i})}\mid 1\leqq i\leqq n\}$; it is evident that $\bigcup\overline{U_{\alpha i}}\supset f(X)$.

For subsets of H-closed spaces and H-closed subsets of a space, we have the following theorems.

6. Theorem H-closed subspace of a Hausdorff space is closed, but not necessarily r.c.

7. Theorem R.c. subset F of a H-closed space Y is H-closed.

proof. Let $\mathfrak{U}$ be an open covering of F. Then $\mathfrak{U}\cup\{F^c\}$ is an open covering of Y, hence reduced to a finite closure covering, i.e.,

$$\bigcup\{\overline{U}_i\mid U_i\in\mathfrak{U},\ 1\leqq i\leqq n\}\cup F^{c-}=X.$$

We have $\bigcup\overline{U}_i\supset F^{c-0}=F^0$ and $\bigcup\overline{U}_i\supset F^{0-}=F$.

8. Theorem Finite union of H-closed subsets is also H-closed.

For compact spaces, we have

9. Theorem ([3], p. 226) Let X be compact, Y be Hausdorff, and $f: X\to Y$ continuous. Then:

(1) f is a closed map.

(2) If f is a bijection, then f is a homeomorphism.

For H-closed spaces, this cannot be true

10. Example Let X be the unit interval of R^1 with the topology $\mathfrak{T}$ generated by the open intervals and Q (the rationals)

(a) $\mathfrak{T}$ is Hausdorff

∵ $\mathfrak{T}$ is finer than the usual topology.

(b) $\mathfrak{T}$ is not regular, hence is not compact.

∵ $X\setminus Q$ is closed in $\mathfrak{T}$ and $0\notin X\setminus Q$.

But $X\setminus Q$ and 0 cannot be separated by disjoint open sets.

(c) $\mathfrak{T}$ is H-closed.

∵ Suppose $E\,\mathfrak{U}=\{U_\alpha\mid\alpha\in A\}$ (open cover of X) which contains no finite subcollection of $\overline{\mathfrak{U}}=\{\overline{U}_\alpha\mid\alpha\in A\}$ covers X. Then one of $\left[0,\ \frac{1}{2}\right]$ and $\left[\frac{1}{2},\ 1\right]$ is not covered by finite subcollection of $\overline{\mathfrak{U}}$. Proceed this subdivision. We have a sequence $\{I_n\}$ with the following properties:

(a) $X=I_1\supset I_2\supset I_3\supset\cdots\cdots$

(b) I_n is not covered by any finite subcollection of $\overline{V}$.

(c) $x, y\in I_n\Rightarrow|x-y|\leqq 2^{-n}$.

Then there is an $x^*\in\bigcap^\infty I_n$.

Let $x^*\in U_\alpha\in\mathfrak{U}$. Since U_α is open in $\mathfrak{T}$, there exists a read $r>0$ such that $(x^*-r,\ x^*+r)\cap Q$

$\cap X \in U_\alpha$. If n is large so enough that $2^{-n} < r$, $I_n \subset [x^* - r,\ x[+r] \cap X = \overline{(x^* - r,\ x^* + r) \cap Q \cap X} \subset \overline{U}_\alpha$.

This contradiction completes the proof.

Now let Y be the unit interval with the usual topology, $f: X \to Y$ be the identity map. Then f is a continuous bijection which is neither closed nor open map. The above example shows that closed subset of a H-closed space is not necessarily H-closed. Consider $X \setminus Q$ in X. (cf. 7 Theorem)

Bibliography

[1] M. Katetov, On H-closed extensions of topological spaces, Caspis matem, fys. 72 (1947), pp. 17—32.

[2] L. Mioduszewski and L. Rudolf, H-closed and extremally disconnected Hausdorff spaces, Dissert. Math. (Roz. Mat.), Warszawa, 1969.

[3] J. Dugundji, Topology, Allyn and Bacon, Inc. 1965.

[4] P.S. Alexandroff et P.S. Urysohn, Mémoire sur les espaces topologiques compacts, Verh. K. Akademic Amsterdam, Deel XIV, Nr 1 (1929), pp. 1—96.

P. 21에서 계속

Bibliography

[1] Walter Rudin, *Principles of mathematical analysis*, McGraw-Hill Book Company 1964.

[2] James Dugundji, *Topology*; Allyn and Bacon in C.Boston 1966.

[3] John L. Kelley, *General topology*; D. Van Nostrand Company, Inc. 1957.

[4] Seymour Lipschutz, *General topology*; Schaum Publishing Co. New York 1965.

[5] Eduin Hewit and Karl Stromberg, *Real and Abstract Analysis*; Springer-Verlag Heidelberg 1969.

(Korea University)

p. 34 에서 계속

의 용어나 상호 관계를 명확히 다루어야 한다.

11. 측도의 영역을 지도함에 있어서는 미터법의 단위 관계의 테두리에서 형식적인 측정기능에 치우치는 것보다 측도의 개념을 인식시켜 근사값과 오차의 인식을 통하여 실용성을 높이도록 다루어야 한다.

(1971. 1)