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Weinberg방정식으로 부터 Faddeev형 방정식의 유도*

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Derivation of Faddeev-Type Equation from Weinberg's Equation*

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요 약 다체(4. 이상)계의 Faddeev형 방정식을 Weinberg방정식으로 부터 다른 저자들에 의한 종전의 방법보다 훨씬 간단하게 유도하였다. 유도된 Faddeev형 방정식을 matrix로 표현 하였고 matrix적분방정식의 matrix kernel과 inhomogeneous term을 구성하는 방법을 규칙화 하였다. 3, 4, 5 체계를 예로 들어서 얻어진 귀칙들을 실증하였다.

Abstract The Faddeev-type equations for systems of more than four particles are derived from weinberg's equation. The derivation is considerably simpler than that by others. The Faddeev-type equations thus derived can be expressed in a matrix form and the rules for constructing the inhomogeneous term and the matrix kernel of the matrix integral equation are formulated and verified explicitly for N=3, 4, and 5.

Introduction

The N-particle Faddeev-type equation is very useful for certain discussions of many-body problems, *e.g.*, theory of transport processes[1].

Such an equation for a four-body system was derived by several authors [2,3,4,5] by extending Faddeev's[6] original idea. Their derivations were invariably complicated. Rosenberg[7] and Yakubovskii[8] also obtained the N-particle extension of Faddeev's equation. In the former's derivation the actual structure of the kernels is obscure and in the latter's derivation it is rather high-browed mathematically and prohibitively formal.

Weinberg[9] obtained an equation for an N-particle system that has only the connected diagrams in the kernel so that the iterative solution would not lead to a divergence difficulty. Although formally correct, Weinberg's equation is not as convenient as Faddeev's equation even for three-particles.

In this note we present a derivation of Faddeev-type equations for systems of particles for $N \geqslant 4$ from Weinberg's equation. It turns out that the derivation of such equations from Weinberg's equation is far simpler than other derivations.

Since the Faddeev-type equation to be derived is a set of coupled equations, it is useful to cast the set in matrix form. For this purpose it is necessary to define a matrix kernel of the

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matrix integral equation. For the case of $N=3$ it is rather simple to obtain the matrix kernel; Its diagonal components are zero while the off-diagonal components are twobody T-matrices in off-energy shell. For $N \geqslant 4$ the structure of the matrix kernel is not so obvious and simple to visualize, since the equations become progressively complicated as N increases. Because of the obvious usefulness of the matrix form of the equations for investigation of some approximation schemes that may be attempted for systems of more than four particles in future, we formulate, based on Weinberg's equation, a set of rules to construct the elements of the matrix kernels for $N \geqslant 4$. These rules make the construction of the kernel a simple mechanical task. Thus formulated rules will be verified for $N=3, 4,$ and 5.

Derivation of Faddeev-type Equation

It is necessary to develop notations in order to facilitate the derivation of the Faddeev-type equations for $N \geqslant 4$. In many-body problems it is useful to express Green's functions (operators) G_s in terms of cluster functions C_s where S denotes the set of particles $1, 2, \cdots N$,

$$G_s(z) = \sum_{m=1}^{N} \frac{1}{m!} \sum_{(S_m)}^{(S)} C_{S_1}(z) * C_{S_2}(z) * \cdots * C_{S_m}(z), \quad \text{(II. 1)}$$

where z is the complex energy and $S_1, S_2, \cdots S_m$ a set of particle indeces of the clusters the union of which is S, *i. e.*, $S = S_1 \cup S_2 \cup \cdots \cup S_m$. The asterisk *denotes the convolution of two or more disconnected clusters C_{S_1}, C_{S_2}, etc. For example, if we denote the particles by $i, j, k, \cdots$ then

$$\begin{aligned}
G_i &= C_i' \\
G_{ij} &= C_{ij} + C_i * C_j' \\
G_{ijk} &= C_{ijk} + C_{ij} * C_k + C_{ik} * C_j + C_{jk} * C_i \\
G_{ijkl} &= C_{ijkl} + C_{ijk} * C_l + C_{ijl} * C_k + {}_{ikl} * C_j + C_{jkl} * C_i \\
&\quad + C_{ij} * C_{kl} + C_{ik} * C_{jl} + C_{il} * C_{jk} \\
&\quad + C_{ij} * C_k * C_l + C_{ik} * C_j * C_l + C_{il} * C_j * C_k \\
&\quad + C_{jk} * C_i * C_l + C_{jl} * C_i * C_k + C_{kl} * C_i * C_j \\
&\quad + C_i * C_j * C_k * C_l, \qquad \text{(II. 2)}
\end{aligned}$$

etc.

By inverting these relations (II. 2), we obtain C_S in terms of G_S,

$$C_S(z) = \sum_{m=1}^{N} \frac{(-1)^{m-1}}{m} \sum_{(S_m)}^{(S)} G_{S_1 S_2 \cdots S_m}(z), \quad \text{(II. 3)}$$

where

$$G_{S_1 S_2 \cdots S_m}(z) = (z - H_{S_1} - H_{S_2} - \cdots - H_{S_m})^{-1}, \quad \text{(II. 4)}$$

with H_{S_i} defined by the Hamiltonian of the cluster i. The irreducible kernel $I_S(z)$ is defined by

$$I_S(z) = \frac{1}{2} \sum_{S'S''}^{(S)} C_{S'}(z) * C_{S''}(z) V_{S'S''}, \quad \text{(II. 5)}$$

where the sum runs over all possible ways of dividing the systrm S into two disjoint clusters S' and S'', and $V_{S'S''}$ is the sum of all V_{ij}'s, the two-body potentials, with the particle i in S' and the particle j in S''. Thus two disjoint clusters $C_{S'}$ and $C_{S''}$ are linked together by the potentials in $V_{S'S''}$ to form connected diagrams. With $I_S(z)$ thus defined, the cluster $C_S(z)$ can be expressed as follows:

$$C_S(z) = I_S(z) G_S(z), \quad \text{(II. 6)}$$

which is Weinberg's equation. On substitution of $G_S(z)$ in the form of (II. 1) into (II. 6), we can obtain an integral equation for $C_S(z)$ with the kernel consisting of connected diagrams only. This equation is called Weinberg's integral equation.

Now we define the T-matrix for a group of particles imbedded in free particles as follows:

$$G_{ij}(z) = G_o(z) + G_o(z) T_{ij}(z) G_o(z), \quad \text{(II. 7)}$$

$$G_{ijk}(z) = G_o(z) + G_o(z) T_{ijk}(z) G_o(z), \quad \text{(II. 8)}$$

$$\vdots$$

$$G_S(z) = G_o(z) + G_o(z) T_S(z) G_o(z). \quad \text{(II. 9)}$$

Here

$$G_o(z)=(z-\sum_{i=1}^{N}H_i)^{-1}, \qquad (\text{II. }9)$$

with H_i defined by a single particle Hamiltonian. Therefore $G_o(z)$ is the Green's function for N free particles and hence the T-matrices are defined in the N particle space.

It is now important to note that the cluster functions may be given in terms of irreducible T-matrix, $\mathcal{T}_{S'}$ as follows:

$$C_{ij}=G_oT_{ij}G_o; \quad T_{ij}\equiv\mathcal{T}_{ij} \qquad (\text{II. }10)$$

$$C_{ijk}=G_o\mathcal{T}_{ijk}G_o, \qquad (\text{II. }11)$$

$$\vdots$$

$$C_S=G_o\mathcal{T}_SG_o, \qquad (\text{II. }12)$$

where

$$\mathcal{T}_{ijk}=T_{ijk}-T_{ij}-T_{ik}-T_{jk}, \qquad (\text{II. }13)$$

$$\vdots$$

$$\mathcal{T}_S=T_S\sum_{m=2}^{N}\frac{1}{m}\sum_{(S_m)}^{(S)}\mathcal{T}_{S_1S_2\cdots S_m} \qquad (\text{II. }14)$$

where the sum runs over all ways of dividing the system S into m disjoint groups of particles. We have divided the sum with m since m different arrangements of $S_1'S_2'\cdots S_m$ give rise to the same irreducible T-matrix. The operator $\mathcal{T}_{S_1S_2\cdots S_m}$ is defined by the equality

$$C_{S_1}*C_{S_2}*\cdots*C_{S_m}=G_0\mathcal{T}_{S_1S_2\cdots S_m}G_o. \quad (11.5)$$

For example,

$$C_{12}*C_3=G_o\mathcal{T}_{12;3}G_o; \ \mathcal{T}_{12;3}=T_{12},$$

and

$$C_{12}*C_{34}=G_o\mathcal{T}_{12;34}G_o,$$

where

$$\mathcal{T}_{12;34}=T_{12;34}-T_{12}-T_{34}.$$

Since

$$V_SG_S=T_SG_o,$$

and V_S is a sum of the pair potentials of the system S, we may define $T_S^{(ij)}$ such that

$$V_{ij}G_S=T_S^{(ij)}G_o.$$

Then we can write

$$V_{S'S''}G_S=\sum_{\alpha}T_S^{(\alpha)}G_o,$$

where the sum runs over all $\alpha=(ij)$ such that $i\epsilon S'$ and $j\epsilon S''$. The irreducible T-matrices $\mathcal{T}_S$ and $\mathcal{T}_{S'S''}$ may be given by

$$\mathcal{T}_S=\sum_{\alpha}\mathcal{T}_S^{(\alpha)} \qquad (\text{II. }16)$$

and

$$\mathcal{T}_{S'S''}=\sum_{\alpha}\mathcal{T}_{S'S''}^{(\alpha)} \qquad (\text{II. }17)$$

When (II. 10)—(II. 17) are made use of, a set of coupled equations for $\mathcal{T}_S^{(\alpha)}$ can be obtained,

$$\mathcal{T}_S^{(\alpha)}=\frac{1}{2}\sum_{S'S''}\delta(\alpha;S',S'')\mathcal{T}_{S'S''}^{(\alpha)}G_o\sum_{\beta\neq\alpha}T_S^{(\beta)} \qquad (\text{II. }18)$$

where

$$\delta(\alpha;S',S'')=\begin{cases}1 & \text{if } \alpha\epsilon S' \text{ or } \alpha\epsilon S'' \\ 0 & \text{otherwise.}\end{cases} \qquad (\text{II. }19)$$

Since

$$\mathcal{T}_S^{(\alpha)}=T_S^{(\alpha)}-\sum_{m=2}^{N-1}\frac{1}{m}\sum_{(S_m)}^{(S)}\delta(\alpha;S_1,S_2,\cdots,S_m)\mathcal{T}_{S_1S_2\cdots S_m}^{(\alpha)}, \qquad (\text{II. }20)$$

where

$$\delta(\alpha;S_1,S_2,\cdots,S_m)=\begin{cases}1 & \text{if } \alpha\in S_1 \text{ or } S_2\cdots\text{or } S_m, \\ 0 & \text{otherwise,}\end{cases}$$

we obtain the following coupled equations for $T_S^{(\alpha)}$,

$$T_S^{(\alpha)}=\sum_{m=1}^{N-1}\frac{1}{m}\sum_{(S_m)}^{(S)}\delta(\alpha;S_1S_2\cdots S_m)\mathcal{T}^{(\alpha)}_{S_1S_2\cdots S_m}$$
$$+\frac{1}{2}\sum_{S'S''}\delta(\alpha;S',S'')\mathcal{T}_{S'S''}^{(\alpha)}G_o\sum_{\beta\neq\alpha}T_S^{(\beta)}, \qquad (\text{II. }21)$$

which is the Faddeev-type equation for the system $S=(1,2,\cdots'N)$. This is a set of $N(N-1)/2$ coupled equations. It is convenient for various reasons to cast (II. 21) into a matrix form. For this purpose we define column vectors of order $N(N-1)/2$,

$$T_S=\{T_S^{(12)},\ T_S^{(13)},\ \cdots,\ T_S^{(23)},\ \cdots,\ T_S^{(N-1,N)}\}, \qquad (\text{II. }22)$$

$$T_{SD}=\{\sum_{m=1}^{N-1}\frac{1}{m}\sum_{(S_m)}^{(S)}\delta(\alpha;S_1,S_2,\cdots,S_m)\mathcal{T}^{(\alpha)}_{S_1S_2\cdots S_m},\cdots\},\tag{II. 23}$$

and $\frac{1}{2}N(N-1)\times\frac{1}{2}N(N-1)$ square matrix M such that

$$M=\begin{bmatrix}0 & M(12\cdots N|ij;kl)\\ M(12\cdots N|kl;ij) & 0\end{bmatrix}\tag{II. 24}$$

The matrix M has no diagonal components and $M(12\cdots N|ij;kl)$ is the $(\alpha\beta)$ element $(\alpha=ij,\ \beta=kl)$ pertaining the particle pairs ij and kl. A rule of constructing the element $M(12\cdots N|ij;kl)$ will be given shortly. With the definitions of (II. 22)—(II. 24) we can express (II. 21) in a matrix form.

$$T_S=T_{SD}+MG_oT_S.\tag{II. 25}$$

We now formulate the rules of constructing (II. 25) without proof. It is however, easy to verify their correctness by taking $N=3,4,5$ and comparing them with the equation obtained directly from (II. 21).

Rule 1. The inhomogeneous term $T_{SD}{}^{(ij)}$ is given by the sum of all the possible (ij) components of $\mathcal{T}$ operators for the disjoint systems $S_1, S_2, \cdots S_m$ *of* S such that the pair (ij) is contained in one and only one subsystem S_m'

$$T_{SD}{}^{(ij)}=\sum_{m=2}^{N-1}\frac{1}{m}\sum_{(S_m)}^{(S)}\mathcal{T}^{(ij)}_{S_1S_2\cdots S_m}.\tag{II. 26}$$

Rule 2. Let i, j, p and q be the particle index, respectively. Then the elements of the matrix kernel M are given by

$$M(12\cdots ij\cdots pq\cdots N|ij;pq)=\frac{1}{2}\sum_{S_1S_2}\mathcal{T}^{(ij)}_{S_1S_2}\tag{II. 27}$$

where the sum runs over (ij) components of $\mathcal{T}$ operators for all the possible two disjoint clusters obtained from the system S by dividing it as follows: Let $S_1 \cup S_2 = S$.

(a) If either one of p *and* q are equal to either one of i and j, *e.g.*, if $i=p$ then it must be that $i,\ j\in S_1$, but $q\in S_2$ or $i,\ j\in S_2$, but $q\in S_1$.

(b) If neither one of p and q are equal to i and j, then it must be that $i, j,\ p\in S_1$, but $q\in S_2$ or $i, j,\ q\in S_1$, but $p\in S_2$ and vice versa.

The case in which $p\in S_2$ and $q\in S_2$ or $p\in S_1$ and $q\in S_1$ simultaneously is not allowed. The factor 1/2 in (II. 26) is there to account for the fact that $\mathcal{T}^{(ij)}_{S_1S_2}=\mathcal{T}^{(ij)}_{S_2S_1}$.

It is easy to verify these rules by constructing the Faddeev and Faddeev-type equations for $N=3, 4$, and 5, rerpectively, and comparing them with those derived by alternative method.

For $N=3$, there are only three ways to divide the particles 1, 2, and 3 into two disjoint clusters. That is, (12) (3), (1) (23), and (13) (2). Therefore by the rule 1 we obtain

$$T^{(ij)}_{SD}=\mathcal{T}^{(ij)}_{ij;k}=T_{ij},\quad (i,\ j, k=1, 2, 3).$$

According to the rule 2, we obtain

$$M(123|ij;jk)=\mathcal{T}_{ij;k}^{(ij)}=T_{ij}$$

$$M(123|ij;ik)=\mathcal{T}_{ij;j}^{(ij)}=T_{ij}$$

and thus the matrix M may be written

$$M=\begin{pmatrix}O & T_{12} & T_{12}\\ T_{13} & O & T_{13}\\ T_{23} & T_{23} & O\end{pmatrix}$$

This is the kernel of the Faddeev equation for $N=3$.

For $N=4$, the particles 1, 2, 3, and 4 may be grouped as follows: (123) (4), (124) (3), (134) (2), (234) (1), (12) (34), (13) (24), (14) (23), (12) (3) (4), (13) (2) (4), (14) (2) (3), (23) (1) (4), (24) (1) (3), (34) (1) (2). Therefore according to the rule 1 we obtain

$$T^{(ij)}_{SD}=T_{ij}+\mathcal{T}^{(ij)}_{ij;kl}+\mathcal{T}^{(ij)}_{ijk;l}+\mathcal{T}^{(ij)}_{ijl;k}$$

where i, j, k, l are the particle indeces and used cyclically. There are 7 ways of dividing the four particles into two disjoint groups; (123)

(4), (124) (3), (134) (2), (234) (1), (12) (34), (13) (24), (14) (23). Therefore the elements of the matrix kernel are given by

$$M(1234|ij;ik)=\mathcal{T}^{(ij)}_{ij;k}+\mathcal{T}^{(ij)}_{ij;kl}\quad(l\neq i,j,k)$$

$$M(1234|ij;kl)=\mathcal{T}^{(ij)}_{ij;k;l}+\mathcal{T}^{(ij)}_{ij;kl}\quad(i,\ j\neq k,l)$$

This kernel agrees with that constructed from the Faddeev-type equation for $N=4$ obtained by Alessandrini [4] Mishma et al [5] Mitra et al [2] and Weyer [3]

For $N=5$, the inhomogeneous terms of the matrix equation (II. 25) are

$$\begin{aligned}T_{3D}^{(ij)}=T_{ij}&+\mathcal{T}^{(ij)}_{ij;kl;m}+\mathcal{T}^{(ij)}_{ij;km;l}+\mathcal{T}^{(ij)}_{ij;k;lm}\\&+\mathcal{T}^{(ij)}_{ij;kl;m}+\mathcal{T}^{(ij)}_{ij;l;km}+\mathcal{T}^{(ij)}_{ij;m;kl}\\&+\mathcal{T}^{(ij)}_{ij;klm}+\mathcal{T}^{(ij)}_{ij;km;l}+\mathcal{T}^{(ij)}_{ij;lm;k}\end{aligned}$$

$$(\text{ijklm}=1,2,3,4,5)\qquad(\text{II. }28)$$

and the elements of the matrix kernel are

$$\begin{aligned}M(12345|ij;kl)=&\mathcal{T}^{(ij)}_{ij;km;l}+\mathcal{T}^{(ij)}_{ij;lm;k}\\&+\mathcal{T}^{(ij)}_{ij;k;lm}+\mathcal{T}^{(ij)}_{ij;l;km},\\&(i,j\neq k,l)\end{aligned}$$

$$\begin{aligned}M(12345|ij;ik)=&\mathcal{T}^{(ij)}_{ij;lm;k}+\mathcal{T}^{(ij)}_{ij;l;mk}\\&+\mathcal{T}^{(ij)}_{ij;m;kl}+\mathcal{T}^{(ij)}_{ij;mkl}\quad(j\neq k).\end{aligned}$$

$$(\text{II. }29)$$

It is easy, although tedious, to check (II. 28) and (II. 29) against the result obtained directly from Weinberg's equation. It is found that they agree with each other.

In summary, we have derived the Faddeev-type equation for N-particle system form Weinberg's equation. Owing to the heuristic and lucidity of Weinberg's derivation of his equation, the present derivation of the Faddeev-type equation is considerably simpler and easier to understand than other derivations.

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