Journal of the Korean Nuclear Society
Volume 4, Number 2, June, 1972

≪Original≫

Time-Dependent Neutron Transport Equation with Delayed Neutrons

Kun Joong Yoo and Pong Youl Pac

Department of Nuclear Engineering, College of Engineering
Seoul National University, Seoul, Korea

(Received March 30, 1972)

Abstract

Time-dependent neutron transport equation with delayed neutrons is analytically solved in the case of isotropic scattering with constant cross sections. The equations in the two divided time regions are obtained from the original equation by the asymptotic method. It is shown that the approximate solutions in each time region are uniformly valid in time to the order of the inverse magnitude of the velocity.

요 약

등방성이고 단면적이 상수인 경우의 지발 중성자를 가진 시간 종속 중성자 수송 방정식이 해석적으로 풀어지고 있다. 두 개로 구분된 시간 영역에 있어서의 방정식이 점근적 방법에 의하여 원래의 수송 방정식으로부터 얻어 지고 있다. 각 시간 영역에 있어서의 근사해는 중성자 속도의 역수 정도로 시간에 있어서 균일하게 유용하다는 것이 보여 지고 있다.

1. Introduction

Time-dependent neutron transport equation has been treated by various authors such as Keepin[1], K.M. Case and P.F. Zweifel[2]. In recent years Hokee Minn solved the time-energy dependent transport equation without delayed neutron[3]. Hendry and Bell showed the time-dependent neuton transport equation with delayed neutron could be solved numerically[4].

If one considers a subcritical system that is exposed to an instantaneous pulse of neutrons, then physically it is clear that for a short time the neutron population will consist almost entirely of neutrons which either were originally injected or one related to original neutrons by prompt events such as fission with the emission of prompt neutrons only. This neutron population, which would result in the absence of decay of delayed neutron precursors, we may call the "prompt pulse". The prompt pulse will rapidly decay, with some prompt rate, $-\alpha_p$, after initial transients. We expect that for samples of physical interest, the prompt decay rate will be very large compared to precursor decay constants and that, therefore, the prompt pulse will have died out before many of the precursors(which are formed in fissions during the prompt pulse) have decayed. Therefore, if one is interested only in the neutron population at late times, it is unnecessary to know the time dependence of the prompt pulse-only the gross production of precursors therein is obtained. In this paper we divided the time region in two parts, where one is the prompt neutron dominant part and the other the delayed neutron dominant part.

Next section Boltzmann transport equation held in each time region will be obtained by the asymptotic expansion. In section 3, we shall seek the solution of the above equation. In the final section, we will discuss the results and offer several remarks.

2. Asymptoic Method

If we consider the case of isotropic scattering, and take the constant cross-sections, the neutron distribution function $\Psi(x,\mu,t)$ and precursor densities $C(x,t)$ satisfy the following equations[1]:

$$\frac{1}{v}\frac{\partial\Psi(x,\mu,t)}{\partial t}+\mu\frac{\partial\Psi(x,\mu,t)}{\partial x}+\Psi(x,\mu,t)=\frac{c(1-\beta)}{2}\int_{-1}^{1}\Psi(x,\mu',t)d\mu'+\lambda C(x,t), \quad (1)$$

and

$$\left(\frac{\partial}{\partial t}+\lambda\right)C(x,t)=\frac{c\beta}{2}\int_{-1}^{1}\Psi(x,\mu',t)d\mu', \quad (2)$$

where μ is $\boldsymbol{v}\cdot\boldsymbol{x}$, $\boldsymbol{v}$ and $\boldsymbol{x}$ are the unit vectors of the neutron velocity and the space respectively, β represents the delayed neutron fraction, λ is the decay constant of delayed neutrons and c is the fission cross section. We may write Eqs. (1) and (2) as

$$\left(\frac{1}{v}\frac{\partial}{\partial t}+B\right)\Psi(x,\mu,t)=\lambda C(x,t), \quad (3)$$

and

$$\left(\frac{\partial}{\partial t}+\lambda\right)C(x,t)=\frac{c\beta}{2}(K\Psi)(x,t), \quad (4)$$

where we have introduced the operators B and K defined such that

$$(B\Psi)(x,\mu,t)\equiv\mu\frac{\partial\Psi(x,\mu,t)}{\partial x}+\Psi(x,\mu,t)-\frac{c(1-\beta)}{2}\int_{-1}^{1}\Psi(x,\mu',t)d\mu'$$

and

$$(K\Psi)(x,t)\equiv\int_{-1}^{1}\Psi(x,\mu',t)d\mu'$$

We are primarily interested in fast systems and the number max $\left(\frac{1}{v}\right)$ is very small. Denoting $\frac{1}{v}$ by ϵ to keep track of the term, Eq. (3) is written as

$$\left(\epsilon\frac{\partial}{\partial t}+B\right)\Psi(x,\mu,t)=\lambda C(x,t) \quad (5)$$

in conjuction with Eq. (4).

The initial conditions will be written as

$$\Psi(x,\mu,o)=\delta(x)\delta(\mu-\mu_o) \quad (6)$$

and

$$C(x,o)=0 \quad (7)$$

We intend to solve Eqs. (4) and (5) by perturbation theory. The idea of the method is that following an initial transient, the time rate of change of Ψ should be of the same order as Ψ, and the solution to Eq. (5) should be well approximated by the solution to the same equation in which the term is omitted.

There are two time ranges of interest: an "inner" range of small t, where the initial conditions on Ψ and C must be satisfied: an "outer" range where the prompt neutron pulse has died away and the solution fed by delayed neutron predominates.

Our approach will be to obtain solutions asymptotically valid in each of the two regions.

First, we examined the inner problem. Since we wish to solve Eq. (5) subject to an initial condition, it will be inappropriate to neglect the derivative term. The contribution of this term can be emphasized by "stretching" the time coordinate. A suitable stretched time will prove to be $\tau=t/\epsilon$. Substituting this into Eqs. (5) and (4), and denoting "inner" by a superscript I, we have

$$\left(\frac{\partial}{\partial\tau}+B\right)\Psi^I(x,\mu,\tau)=\lambda C^I(x,\tau), \tag{8}$$

and

$$\left(\frac{\partial}{\partial\tau}+\epsilon\lambda\right)C^I(x,t)=\frac{\epsilon c\beta}{2}(K\Psi^I)(x,\tau). \tag{9}$$

To solve these equations, we introduce the expansions

$$\Psi^I(x,\mu,\tau)=\Psi^{0I}(x,\mu,\tau)+\epsilon\Psi^{1I}+O(\epsilon^2), \tag{10}$$

and

$$C^I(x,\tau)=C^{0I}(x,\tau)+\epsilon C^{1I}+O(\epsilon^2) \tag{11}$$

Substituting and equating equal powers of ϵ, we obtain

$$\left(\frac{\partial}{\partial\tau}+B\right)\Psi^{0I}(x,\mu,\tau)=\lambda C^{0I}(x,\tau), \tag{12}$$

$$\left(\frac{\partial}{\partial\tau}+B\right)\Psi^{1I}(x,\mu,\tau)=\lambda C^{1I}(x,\tau), \tag{13}$$

$$\frac{\partial C^{0I}(x,\tau)}{\partial\tau}=0, \tag{14}$$

and

$$\frac{\partial C^{1I}(x,\tau)}{\partial\tau}=\frac{c\beta}{2}(K\Psi^{0I})(x,\tau)-\lambda C^{0I}(x,\tau). \tag{15}$$

These are to be solved subject to exact boundary conditions and the initial conditions

$$\Psi^{0I}(x,\mu,o)=\delta(x)\ \delta(\mu-\mu_o), \tag{16}$$

$$\Psi^{1I}(x,\mu,o)=0, \tag{17}$$

and

$$C^{0I}(x,o)=0. \tag{18}$$

Eqs. (14) and (15) may be solved explicitly:

$$C^{0I}(x,\tau)=0, \tag{19}$$

and

$$C^{1I}(x,\tau)=\frac{c\beta}{2}\int_o^\tau d\tau(K\Psi^{0I})(x,\tau). \tag{20}$$

Then Eqs. (12) and (13) become

$$\left(\frac{\partial}{\partial\tau}+B\right)\Psi^{0I}(x,\mu,\tau)=0, \tag{21}$$

and

$$\left(\frac{\partial}{\partial\tau}+B\right)\Psi^{1I}(x,\mu,\tau)=\frac{\lambda c\beta}{2}\int_o^\tau d\tau'(K\Psi^{0I})(x,\tau'), \tag{22}$$

which are subjected to Eqs. (16) and (17).

We now turn to the outer problem. No stretching is needed here-the expansions

$$\Psi^{II}(x,\mu,t)=\Psi^{0II}(x,\mu,t)+\epsilon\Psi^{1II}(x,\mu,t)+O(\epsilon^2), \tag{23}$$

and

$$C^{II}(x,t)=C^{0II}(x,t)+\epsilon C^{1II}(x,t)+O(\epsilon^2) \tag{24}$$

are substituted into Eqs. (5) and (4), and equal powers of ϵ are equated, with the result

$$(B\Psi^{0II})(x,\mu,t)=\lambda C^{0II}(x,t), \tag{25}$$

$$(B\Psi^{1II})(x,\mu,t)=\lambda C^{1II}(x,t)-\frac{\partial\Psi^{0II}(x,\mu,t)}{\partial t}, \tag{26}$$

and

$$\left(\frac{\partial}{\partial t}+\lambda\right)C^{jII}(x,t)=\frac{c\beta}{2}(K\Psi^{jII})(x,t). \tag{27}$$

Eq. (27) has the solution

$$C^{jII}(x,t)=C^{jII}(x,o)exp(-\lambda t)+\frac{c\beta}{2}\int_o^t exp[-\lambda(t-t')]\times(K\Psi^{jII})(x,t')dt'. \tag{28}$$

The $C^{jII}(x,o)$ remains to be determined. This is done by demanding that the inner solution agree with the outer solution at $\tau=\tau_1$. Thus, from Eq. (16), we conclude that

$$C^{0II}(x,o)=0 \tag{29}$$

and Eq. (20) implies that

$$C^{1II}(x,o)=\frac{c\beta}{2}\int_o^{\tau_1}(K\Psi^{0I})(x,\tau')d\tau'. \tag{30}$$

Combining Eqs. (28) and (30) and substituting into Eq. (26), we obtain

$$(B\Psi^{1II})(x,\mu,t)=\lambda\frac{c\beta}{2}\times\left\{\int_o^{\tau_1}d\tau'(K\Psi^{0I})(x,\tau')exp(-\lambda t)+\int_o^t exp[-\lambda(t-t')](K\Psi^{1II})(x,t')dt'\right\} \tag{31}$$

3. Normal Mode Expansion in Time

First let us find the inner solution. We will introduce the method for solving the following equation:[2)]

$$B\Psi(x,\mu)=0$$

or

$$\mu\frac{\partial\Psi(x,\mu)}{\partial x}+\Psi(x,\mu)=\frac{c(1-\beta)}{2}\int_{-1}^1\Psi(x,\mu')d\mu' \tag{32}$$

This approach is suggested by the usual method of solving ordinary differential equations; *i. e.* the solution is expanded in a series of solutions of the homogeneous equation or normal modes[5].

We seek the solutions of Eq. (1). Translational invariance suggests that we look for solutions of the form

$$\Psi_\nu(x,\mu)=\phi_\nu(\mu)e^{-x/\nu}. \tag{33}$$

Inserting that ansatz into the homogenous equation, we find that

$$\left(1-\frac{\mu}{\nu}\right)\phi_\nu(\mu)=\frac{c}{2}\int_{-1}^{1}\phi_\nu(\mu')d\mu'. \tag{34}$$

Since this is a linear homogeneous equation for ϕ_ν, the normalization is arbitrary. It is convenient to choose

$$\int_{-1}^{1}\phi_\nu(\mu')d\mu'=1. \tag{35}$$

With the normalization Eq. (35), Eq. (34) becomes

$$(\nu-\mu)\phi_\nu(\mu)=\frac{c\nu}{2} \tag{36}$$

Eq. (36) appears to be rather simple, but the essense of the method described here lies in the fact that this equation is not as simple as appears at first glance. Thus it is tempting to write from Eq. (36)

$$\phi_\nu(\mu)=\frac{c\nu}{2}\ \frac{1}{\nu-\mu}. \tag{37}$$

However suppose we allow the possibility that $\mu=\nu$. Then it is clear that we may add to the right-hand side of Eq. (37) a term

$$\lambda(\nu)\delta(\nu-\mu), \tag{38}$$

where λ (ν) is an arbitrary function, and the result still be a solution of Eq. (36). That this is true can be ascertained by substituting the complete result,

$$\phi_\nu(\mu)=\frac{c\nu}{2}\ \frac{1}{\nu-\mu}+\lambda(\nu)\delta(\mu-\nu), \tag{39}$$

directly into Eq. (36) and remembering that by definition

$$x\delta(x)\equiv 0. \tag{40}$$

Thus the final soultion to Eq. (36) becomes

$$\phi_\nu(\mu)=\frac{c\nu}{2}P\frac{1}{\nu-\mu}+\lambda(\nu)\delta(\nu-\mu), \tag{41}$$

where the symbol P to the factor $\frac{1}{\nu-\mu}$ denote the Cauchy principal value.

Suppose that ν does not lie on the real line between -1 and $+1$. Then Eq. (41) becomes simply

$$\phi_\nu(\mu)=\frac{c\nu}{2}\ \frac{1}{\nu-\mu}, \tag{42}$$

and the normalization condition, Eq. (35), gives

$$\Lambda(\nu)\equiv 1-\frac{c\nu}{2}\int_{-1}^{1}\frac{d\mu}{\nu-\mu}=0. \tag{43}$$

We now must find the zeros of $\Lambda(\nu)$. The following easily verified properties of $\Lambda(\nu)$ will be helpful.

1) $\Lambda(\nu)=\Lambda(-\nu)$. Then if ν_o is a root of Eq. (43), $-\nu_o$ is a root as well.

2) We note from Eq. (43) that if ν_o is a solution of Eq. (43), then ν_o^* is also a solution.

3) We can show that $\Lambda(\nu)$ has only two zeros in the complex plane cut from -1 to $+1$. Then from (1) and (2) in the above, it follows that the zeros of $\Lambda(\nu)$ must lie on the real or imaginary axes.

Having found the eigenvalues, we obtain the eigenfunctions from Eq. (42). Denoting by $\phi_{o\pm}$ the eigenfunctions associated with $\pm\nu_o$ respectively, we have

$$\phi_{o\pm}(\mu)=\pm c\frac{\nu_o}{2}\ \frac{1}{\pm\nu_o-\mu}. \tag{44}$$

These are the discrete modes of the transport equation.

If ν lies on the real line between -1 and $+1$, then we must use Eq. (41). Applying the normalization condition, we find that

$$\lambda(\nu)=1-\frac{c\nu}{2}P\int_{-1}^{1}\frac{d\mu}{\nu-\mu}. \tag{45}$$

The functions $\phi_\nu(\mu)$ are orthogonal in the following sense:

$$\int_{-1}^{1}\mu\phi_\nu(\mu)\phi_{\nu'}(\mu)d\mu=0 \quad \nu\neq\nu'. \tag{46}$$

For the discrete modes, we have

$$N_{o\pm}\equiv\int_{-1}^{1}\mu\phi_{o\pm}^2(\mu)d\mu. \tag{47}$$

For the continuum modes, we get

$$N(\nu)=\nu\left\{\lambda^2(\nu)+\frac{\pi^2c^2}{4}\nu^2\right\}. \tag{48}$$

Now let us return to Eq. (21). We carry out a Laplace transformation on Eq. (21) by multiplying by $e^{(1-s)\tau}d\tau$ and integrating over τ from 0 to ∞. Eq. (21) reduced as follows,

$$\mu\frac{\partial\Psi_s^{OI}}{\partial x}+s\Psi_s^{OI}(x,\mu)=\frac{c(1-\beta)}{2}\times\int_{-1}^{1}\Psi_s^{OI}(x,\mu')d\mu'+\delta(x)\delta(\mu-\mu_o). \quad (49)$$

The solution of Eq. (49) is obtained by K. M. Case as follows:

$$\Psi^{OI}(x,\mu)=\int_0^{*1}\frac{\phi_{\nu,s}^{OI}(\mu_o)\,\phi_{\nu,s}^{OI}(\mu)}{N_s^{OI}(\nu)}e^{-sx/\nu}+\Big(\frac{\phi_{O-,s}^{OI}(\mu)\,\phi_{O-,s}^{OI}(\mu)}{N_s^{OI}(s)}e^{sx/\nu_o^{OI}}-\frac{\phi_{O+,s}^{OI}(\mu_o)\,\phi_{o+,s}^{OI}(\mu)}{N_{o+}^{OI}(s)}e^{-sx/\nu_o^{OI}}\Big). \quad (50)$$

Therefore,

$$\Psi^{OI}(x,\mu,\tau)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Big(\int_0^{*1}\frac{\phi_{\nu,s}^{OI}(\mu_o)\,\phi_{\nu,s}^{OI}(\mu)}{N_s^{OI}(\nu)}e^{-sx/\nu}d\nu\Big)\times e^{-(1-s)\tau}ds+\frac{1}{2\pi i}\times\int_o^{c(1-\beta)}\Big[\frac{\phi_{o-,s}^{OI}(\mu_o)\,\phi_{o-,s}^{OI}(\mu)}{N_{o-}^{OI}(s)}e^{sx/\nu_o^{oI}}-\frac{\phi_{o+,s}^{OI}(\mu_o)\,\phi_{o+,s}^{OI}(\mu)}{N_{o+}^{OI}(s)}e^{-sx/\nu_o^{oI}}\Big]e^{-(1-s)\tau}ds \quad (51)$$

where $a_{s\pm}^{OI}=\phi_{o+,s}^{OI}(\mu_o)/N_{o\pm}^{OI}(s)$ (52)

$$N_{o\pm}^{OI}(s)=\frac{c(1-\beta)(\nu_o^{OI})^2}{2s}\times\Big(\frac{c(1-\beta)}{2s}\frac{\nu_o^{OI}}{(\nu_o^{OI})^2-1}-\frac{1}{\nu_o^{OI}}\Big)=\pm\nu_o^{OI}\frac{d\Lambda}{d\nu}\Big|_{\nu=\nu_o^{OI}} \quad (53)$$

and

$$N_s^{OI}(\nu)=\nu[(\lambda_s^{OI})^2(\nu)+\{\pi c(1-\beta)\nu/2s\}^2]. \quad (54)$$

Carrying out a Laplace transformation on Eq. (22) to find the solution of the equation, we get the following equation:

$$\mu\frac{\partial\Psi_s^{1I}(x,\mu)}{\partial x}+s\Psi_s^{1I}(x,\mu)=\frac{c(1-\beta)}{2}\int_{-1}^{1}\Psi_s^{1I}(x,\mu')d\mu'-\frac{\lambda c\beta}{2(1-s)}\int_{-1}^{1}\Psi_o^{OI}(x,\mu',\tau')d\mu'. \quad (55)$$

The solution of Eq. (55) may be obtained from the following equation:

$$\mu\frac{\partial\Psi_{gs}^{1I}(x,\mu)}{\partial x}+s\Psi_{gs}^{1I}(x,\mu)=\frac{c(1-\beta)}{2}\int_{-1}^{1}\Psi_{gs}^{1I}(\mu')d\mu'+\delta(x-x')\delta(\mu-\mu''). \quad (56)$$

Eq. (56) is equivalent to Eq. (49) if the replacement $x\to x-x'$ and $\mu_o\to\mu''$. Therefore Ψ_{gs}^{1I} is obtained as follows;

$$\Psi_{gs}^{1I}(x,\mu)=\Big(\int_0^{*1}\frac{\phi_{\nu,s}^{O\pm}(\mu'')\,\phi_{\nu,s}^{OI}(\mu)}{N_s^{OI}(\nu)}e^{-s(x-x')sx/\nu}d\nu\Big)+\Big(\frac{\phi_{o-,s}^{OI}(\mu'')\,\phi_{o-,s}^{OI}(\mu)}{N_{o-}^{OI}(s)}e^{s(x-x')\nu/\nu_o^{oI}}-\frac{\phi_{o+,s}^{OI}(\mu'')\,\phi_{o+,s}^{OI}(\mu)}{N_{o+}^{OI}(s)}e^{-s(x-x')\nu/\nu_o^{oI}}\Big) \quad (57)$$

Therefore we obtained $\Psi_s^{1I}(x,\mu)$ as follows;

$$\Psi_s^{1I}(x,\mu)=\int_{-1}^{1}d\mu''\int_{-\infty}^{\infty}dx'\Psi_{gs}^{1I}(x,\mu)\times\Big(-\frac{\lambda c\beta}{2(1-s)}\int_{-1}^{1}\Psi_s^{OI}(x',\mu',\tau')d\mu'\Big), \quad (58)$$

and

$$\Psi^{1I}(x,\mu)=\frac{1}{2\pi i}\int_{\tau_1-i\infty}^{\tau_1+i\infty}\Psi_s^{1I}(x,\mu)e^{-(1-s)\tau}d\tau. \quad (59)$$

So the inner solution is completely obtained.

Finally we will solve the outer solution. Combining Eqs. (28) and (29), we conclude that Eq. (25) is homogeneous in Ψ^{OII}; since Ψ^{OI} as $\tau\to\tau_1$, we have

$$\Psi^{OII}(x,\mu,t)=0. \quad (60)$$

Differentiating the equation multiplied Eq. (31) by $e^{\lambda t}$ and taking the Laplace transform, we get

$$\mu\frac{\partial\Psi_s^{1II}(x,\mu)}{\partial x}+\Psi_s^{1II}(x,\mu)$$
$$=\frac{c}{2}\left[(1-\beta)+\frac{\lambda\beta}{\lambda+s-1}\right]$$
$$\times\int_{-1}^{1}\Psi_s^{1II}(x,\mu')d\mu'+\frac{1}{\lambda+s-1}$$
$$\times\left[\mu\frac{\partial\Psi^{1II}(x,\mu,o)}{\partial x}+\Psi^{1II}(x,\mu,o)\right.$$
$$\left.-\frac{c(1-\beta)}{2}\int_{-1}^{1}\Psi^{1II}(x,\mu')d\mu'\right], \quad (61)$$

where $\Psi^{1II}(x,\mu,o)=\frac{\lambda c\beta}{2}\int_0^{\tau_1}d\tau'(K\Psi^{OI})(x,\tau')$.

We may rewrite Eq. (61) as

$$\mu\frac{\partial\Psi_s^{1II}(x,\mu)}{\partial x}+\Psi_s^{1II}(x,\mu)$$
$$=\frac{c}{2}\left[(1-\beta)+\frac{\lambda\beta}{\lambda+s-1}\right]$$
$$\times\int_{-1}^{1}\Psi_s^{1II}(x,\mu')d\mu'+\frac{1}{\lambda+s-1}\left(\frac{c\lambda\beta}{2}\right)$$
$$\times\int_0^{\tau_1}dt\left[\int_{-1}^{1}d\mu'\left(\mu\frac{\partial\Psi^{OI}(x,\mu,t)}{\partial x}\right.\right.$$
$$+\{1-c(1-\beta)\}\Psi^{OI}(x,\mu,t)\Big)\Big]. \quad (62)$$

First let us solve the following equation to find the solution of Eq. (62).

$$\mu\frac{\partial\Psi_{gs}^{1II}}{\partial x}+\Psi_{gs}^{1II}(x,\mu)=\frac{c}{2}\left[(1-\beta)+\frac{\lambda\beta}{\lambda+s-1}\right]$$
$$\times\int_{-1}^{1}\Psi_{gs}^{1II}(x,\mu')d\mu'+\delta(x-x')\delta(\mu-\mu''). \quad (63)$$

We find that the function $\Psi_{gs}^{1II}(x,\mu)$ obeys the homogeneous equation as the eigenfunctions $\phi_{\nu,s}^{1II}(\mu)e^{-sx/\nu}$ everywhere except at $x=x'$. At that point a discontinuous is introduced by the source:

$$\Psi_{gs}^{1II}(x'+,\mu)-\Psi_{gs}^{1II}(x'-,\mu)=\frac{\delta(\mu-\mu'')}{\mu}. \quad (64)$$

The boundary conditions at $x=\pm\infty$, together with the completeness of the singular eigenfunctions, permit us to write

$$\Psi_{gs}^{1II}(x,\mu)=\int_0^1 A_s^{1II}(\nu)\,\phi_{\nu,s}^{1II}e^{-sx/\nu}d\nu$$
$$+a_{s+}^{1II}\,\phi_{o+,s}^{1II}(\mu)e^{-sx/\nu_o^{1II}},\quad x>x' \quad (65a)$$

and

$$\Psi_{gs}^{1II}(x,\mu)=-\int_{-1}^{0}A_s^{1II}(\nu)\,\phi_{\nu,s}^{1II}\,e^{-sx/\nu}d\nu$$
$$-a_{s-}^{1II}\,\phi_{0-,s}^{1II}(\mu)\,e^{sx/\nu_o^{1II}},\quad x<x' \quad (65b)$$

where we have chosen $\mathrm{Re}(\nu_0^{1II})>0$. The jump condition, Eq (64) determines the coefficients $A_s^{1II}(\nu)$ and $a_{s\pm}^{1II}$:

$$\delta(\mu-\mu_0)/\mu=\int_{-1}^{1}A_s^{1II}(\nu)\,\phi_{\nu,s}^{1II}(\mu)d\nu$$
$$+a_{s+}^{1II}\,\phi_{0+,s}^{1II}(\mu)+a_{s-}^{1II}\,\phi_{0-,s}^{1II}(\mu) \quad (66)$$

Applying the orthogonality relations to Eq. (65) gives

$$a_{s\pm}^{1II}=\phi_{o\pm,s}(\mu'')/N_{0\pm}^{1II}(s), \quad (67a)$$

and

$$A_s^{1II}=\phi_{\nu,s}^{1II}(\mu'')/N_s^{1II}(\nu), \quad (67b)$$

where $N_{0\pm}^{1II}(s)$ and $N_s^{1II}(\nu)$ are

$$N_{0\pm}^{1II}(s)=\int_{-1}^{1}\mu(\phi_{0\pm,s}^{1II})^2d\mu \quad (68a)$$

and

$$N_s^{1II}(\nu)=\nu\left[(\lambda_s^{0II}(\nu))^2\right.$$
$$\left.+\frac{\pi\nu c}{2}\left[(1-\beta)+\frac{\lambda\beta}{\lambda+s-1}\right]^2\right]. \quad (68b)$$

Applying the normalization condition gives us

$$\phi_{\nu,s}^{1II}(x,\mu)=\frac{c\nu}{2(\nu-\mu s)}\left[(1-\beta)+\frac{\lambda\beta}{\lambda+s-1}\right], \quad (69)$$

and

$$\lambda_s^{1II}(\nu)=1-\frac{c\nu}{2}\left[(1-\beta)\right.$$
$$\left.+\frac{\lambda\beta}{\lambda+s-1}\right]P\int_{-1}^{1}\frac{d\mu}{\nu-\mu s}. \quad (70)$$

Thus we have got

$$\Psi_{gs}^{1II}(x,\mu)=\int_0^{\pm s}\frac{\phi_{\nu,s}^{1II}(\mu'')\,\phi_{\nu,s}^{1II}(\mu)}{N_s^{1II}(\nu)}e^{-s(x-x')/\nu}$$
$$+\frac{\phi_{0-,s}^{1II}(\mu'')\,\phi_{0-,s}^{1II}(\mu)}{N_{0-}^{1II}(s)}e^{s(x-x)/\nu_o^{1II}}$$

$$-\frac{\phi_{0+,s}^{1II}(\mu'')\,\phi_{0+,s}^{1II}(\mu)}{N_{0+}^{1II}(s)}e^{-s(x-x')/\nu_0^{1II}} \quad (71)$$

We obtained

$$\Psi_s^{1II}(x,\mu)=\frac{1}{\lambda+s-1}\left(\frac{c\lambda\beta}{2}\right)$$
$$\times\int_{-1}^{1}d\mu'\int_{-\infty}^{\infty}dx'\Psi_{gs}^{1II}(x,\mu)$$
$$\times\left(\int_0^{\tau_1}d\mu'\left\{\mu\frac{\partial\Psi^{0I}(x',\mu')}{\partial x}\right.\right.$$
$$\left.\left.+[1-c(1-\beta)]\Psi^{0I}\right\}\right) \quad (72)$$

Taking the Laplace inversion transform of Eq. (72), we obtain

$$\Psi^{1II}(x,\mu)=\frac{1}{2\pi i}\int_{r_2-i\infty}^{r_2+i\infty}\Psi_s^{1II}(x,\mu)e^{-(1-s)t}ds. \quad (73)$$

In summary, we get

$\Psi^I=\Psi^{0I}+\epsilon\Psi^{1I}, \quad \tau<\tau_1$

and

$\Psi^{II}=\epsilon\Psi^{1II}. \quad t>\epsilon\tau_1$

4. Remarks

In the previous section, we have analytically obtained the asymptotic solutions which are valid in all the time regions to the ϵ— order. As $\beta=0$, the solution of the neutron transport equation with delayed neutrons agrees with the well known solution of the neutron transport equation with prompt neutrons only[2].

References

1) G. R. Keepin, Physics of Nuclear Kinetics, 162 Addison-Wesley Publishing Co., Reading, Massachusetts(1965)

2) K. M. Case and P. F. Zweifel, Linear Transport Theory, Addison-Wesley Publishing Co., Reading, Massachusetts(1967)

3) Hokee Minn, Pulsed Energy Dependent Neutron Transport Theory, Journal of the Korean Nuclear Society, **2** (1970)

4) W. L. Hendry and G. I. Bell, An Analysis of the Time-Dependent Neutron Transport Equation with Delayed Neutrons by the Method of Matched Asymptotic Expansions, Nuclear Science and Engineering, **35**, 240(1969)

5) K. M. Case, Elementary Solutions of the Transport Equation, Ann. Phy., **9**, 1 (1960)