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# ≪Original≫ Computer Simulation of the Computational Method in Fuel Optimal Control

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**Abstract**

Determination of a two-point boundary value problem is the key of finding the control function $u(t)$ with the application of the fundamental idea of Minimum principle. The late development shows the discovery of the initial costate vector for the solution of a two-point value problem. As a new technique of determining the optimal control function, Newton's Sequential method is examined about a number of engineering problems and found available.

**요 약**

최대치(또는 최소치) 원리의 기본개념을 써서 제어함수 $u(t)$를 찾는데는 2점간 경계치 문제의 해결이 요구된다. 그런데 최근에는 2점간 문제를 푸는 방법으로 초기 costate vector를 찾아 문제를 해결하는 여러가지 편법이 개발되고 있다.

여기에서는 최적제어 함수를 찾는 새로운 방법의 하나로써 고안된 Newton's Sequential방법을 써서 여러가지 공학적 문제를 풀어 봄으로써 이 방법이 매우 유효하다는 것을 알아냈다.

## 1. Introduction

As the methods of determining optimal control function $u^*(t)$, the method of functional analysis by Krosovski, Minimum principle by Pontryagin, and Dynamic programming by Bellman have been well introduced as typical ones. The application of Minimum principle focuses on the two-point boundary value problem. This solution of boundary value problem is to be the point of determining the optimal function but Minimum principle has its weakness which provides us with only the necessary condition for detrmining the optimal function.

The lack of the sufficient condition was indicated by Kelly[1], Gottlieb[2], Denn[3], and Kurihari[4]. They furthered the method, in detail, for finding optimal control and presented how to minimize a sample Hamiltonian at each iteration. Newton's method which the author is going to introduce was widely studied by McReynolds & Bryson[5], Plant[6], Knudson[7-13]. However the sequential operation method was not discussed in their papers.

In previous paper[14], the new sequential

operation method applied to Newton's method is designed for determining optimal control function $u^*(t)$.

In this paper, under a suitable assumption, the main part of this method is proved by solving a number of engineering problems by computer.

## 2. Outline of the Computational Method[14]

The fixed time fuel optimal control of a linear plant to a given state as a problem is examined. The problem is presented, reduced to a two-point boundary value problem, changed to integral form, replaced by a sequence of approximate integral equations, and made ready for the numerical solution by the application of Newton's method.

Problem

Given; a. A system described by the linear time invariant(vector) differential equation

$$\dot{X}(t)=AX(t)+bu(t) \tag{2.1}$$

where

1. Then vector $X(t)$ is the state.

2. The system matrix $A$ is an $n\times n$ constant matrix.

3. The gain matrix $b$ is an $n\times r$ constant matrix.

4. The $r$ vector $u(t)$ is the control.

b. A fixed time interval

$$t\epsilon[O, T] \tag{2.2}$$

c. Initial and terminal boundary conditions on the state vector

$$X(O)=\xi$$
$$X(T)=\theta \tag{2.3}$$

d. The control variable must satisfy a constraint

$$|u(t)|\leq 1 \text{ for all } t\ \epsilon[O, T] \tag{2.3}$$

e. The fuel functional is

$$J(u)=\int_o^T |u(t)|dt. \tag{2.4}$$

Then, it is desired to find a control variable $u^*(t)$ that

a. Satisfies the constraint (2.4)

b. Transfers the system (2.1) from the initial state $\xi$ at time $t=O$ to the terminal state $\theta$ at time $t=T$,

c. Minimizes the feul function. (2.5)

The relations deduced by applying Pontryagin's minimum principle to the problem are summarized below;

Definition; The "deadzone" function $dez[-]$ is defined as follows;

$$\text{means} \quad u(t)=dez[w(t)] \tag{2.6}$$
$$u(t)=1 \quad \text{when} \quad w(t)>1$$
$$u(t)=0 \quad \text{when} \quad |w(t)|<1$$
$$u(t)=-1 \quad \text{when} \quad w(t)<-1$$

Let $u^*(t)$, $t\epsilon[O, T]$ be the fuel optimal control, the solution of problem, assuming that one exists. Let $X^*(t)$ be the resulting state on the fuel optimal trajectory. Let $P^*(t)$ $t\epsilon[O, T]$ be the corresponding costate vector.

Then the minimum principle yields the relations

$$H(X^*, u^*, P^*, t)=|u^*(t)|+P^*(t)Ax^*(t)$$
$$+P^*(t)bu^*(t) \tag{2.7}$$

$$\dot{X}^*(t)=\frac{\partial H}{\partial P^*}=AX^*(t)+bu^*(t) \tag{2.8}$$

$$\dot{P}^*(t)=-\frac{\partial H}{\partial X^*}=-A'P^*(t) \tag{2.9}$$

$$X^*(O)=\xi$$
$$X^*(T)=\theta \tag{2.10}$$

where $A'$ is the transpose of $A$

and the relation

$H(X^*, u^*, p^*, t)\leq H(X^*, u, P^*, t)$ for all $u$ such that $|u|\leq 1$ yield $u^*(t)=-dez[b'p^*(t)]$ (2.11) from above *Eqs.* (2.8)-(2.11). Determination of $\pi^*$, the optimal costate initial condition vector, will be considered equivalent to solution of the Two point boundary value problem.

For later use with Newton's method, $q(t)$ and the operator $T(\pi)$ are defined as;

$$q(t)=e^{-A't}\ b \tag{2.12}$$

$$T(\pi)=\xi-e^{-A't}\theta-\int_o^T q(t)\ dez[q'(t)\pi]dt \tag{2.13}$$

For simplicity, $\pi^*$ will refered to as the solution of the operator $T(\pi)$, also, in most of

what follows the final state is the origin, so the integral equation form of $T(\pi)$ becomes

$$T(\pi)=\xi-\int_o^T q(t)\ dez[q'(t)\pi]dt \qquad (2,14)$$

A sequence of approximate operator$\{T_k(\pi)\}$ is now introduced to replace the operator $T(\pi)$ that is;

$$T_k=(\pi)=\xi-\alpha_k W(T)\pi-\int_o^T q(t)\ U_k(q'(t)\pi)dt \qquad (2,15)$$

where $W(T)$ is the controllability matrix

$$W(T)=\int_o^T q(t)q'(t)dt \qquad (2.16)$$

$q'(t)\pi$ is the control argument.

Using the control argument $q'(t)\pi$, an approximate control function $U_k(\cdot)$ yields

$$U_k(q'(t)\pi)=\frac{1}{2}\{tanh[\eta_k(q'(t)\pi+1)] +tanh[\eta_k(q'(t)\pi-1)]\} \qquad (2.17)$$

The idea is to start with a very simple operator and work up by step toward the exact operator $T(\pi)$.

Newton's method is to be applied to a typical operator $T_k(\pi)$. Given the operator $Eq(2.15)$ to find the solution vector $\pi_k$ such that

$T_k(\pi_k)=O$

One linearizes about the current guess

$T_k(\pi_k)\approx T_k(\pi^i)+(\pi_k-\pi^i)T^{(1)}\ (\pi^i)$

Then the next iteration is found by solving this linear equation for $\pi_k$

$$\pi^{i+1}=\pi^i-[T_k^{(1)}(\pi^i)]^{-1}T_k(\pi^i) \qquad (2.18)$$

Equation (2.18) is the recursive relation of Newton's method. Since $T_k$ has vector valued range space, its first derivative is the Jacobian matrix.

$$T_k^{(1)}\ (\pi)=-\alpha_k\ W(T)-\int_o^T q(t)q'(t)u_k^{(1)}[q'(t)\pi]dt$$

Then *Eq.* (2.18) can be written out entirely in matrix notation

$$\pi^{i+1}=\pi^i+\{\alpha_k W(T)+\int_o^T q(t)q'(t)u_k^{(1)}[q'(t)\pi^i]dt\}^{-1}\left\{\xi-\alpha_k W(T)\pi^i-\int_o^T q(t)u_k[q'(t)\pi^i)dt\right\} \qquad (2.19)$$

From *Eq.* (2.17), the first derivative of the approximate control function $u_k^{(1)}$ is

$$u_k^{(1)}[q'(t)\pi]=\frac{1}{2}\eta_k\{2-tanh^2[\eta_k(q'(t)\pi+1)] -tanh^2[\eta_k(q'(t)\pi-1)]\} \qquad (2.20)$$

Starting with an initial guess $\pi^0$, *Eq.* (2.19) is applied repeatedly, If at same step, $i$, $\pi^i\approx\pi^{i-1}$ the inner loop is said to have converged, and the vector $\pi^i$ is defined to be the solution vector $\pi_k$ of the operator $T_k$.

## 3. Program

It was desired to have a program that would be as flexible as possible within the frame work of Problem and at the same time, as easy to use and as comprehensive as possible. For instance, the fundamental matrix is computed directly from its series definition, so that any system matrix can be used.

A flow chart of main program is shown in Fig. 3.1 with the essential portions of the program showing the relation which were used for the computer solution. The program has been broken into units called subroutines. There is a main program which assumes most of the read in-print out and the internal routing responsibilities the various other subroutines are connected to main program and operate in more or less of a sequence. Decisions as to which optimal subroutines to use are made in main program.

The data include the variable $n$, $T$, $\xi$, $\theta$, $A$, $b$ and the decision constants. The decision constants are named EPS, AMAX, EPMTX, ALPT, M, ICHO, and KPETA. These constants were tested in the earlier experiment and then standardized at what appeared to be resonable values. Among constants the value of the control constant ICHO depends on the made of operation desired. The safe method is to use a large value for the constant $M(M\leq 100)$. The parameters of each approximate operator are indicated, and the costate initial

condition vector is included for each step of Newton's method. If anything goes wrong, such as a martrix inversion difficulty, an apropose warning statement is given and appropriate action is taken, *i. e.*, the program stop if this inversion become too difficult numerically.

Finally, main program decides when the program should stop, by comparing the New value of $\eta$ found in subroutine with the given decision constant AMAX.

The program stops when

$$\eta_k > \text{AMAX} \qquad (3.1)$$

Experience shows that the procedure has converged pretty well by the time $\eta=5$ or 10, so as a normal value one uses AMAX=10

## 4. Computer Result

A number of computer runs were made to test the method and to try it out on various example. Some of the more enlightening ones are enumerated in this section, to gether with their purpose and chief result.

The runs are listed by state (or plant) matrix. In each case the Jordan canonical form was used, with the added requirement that all

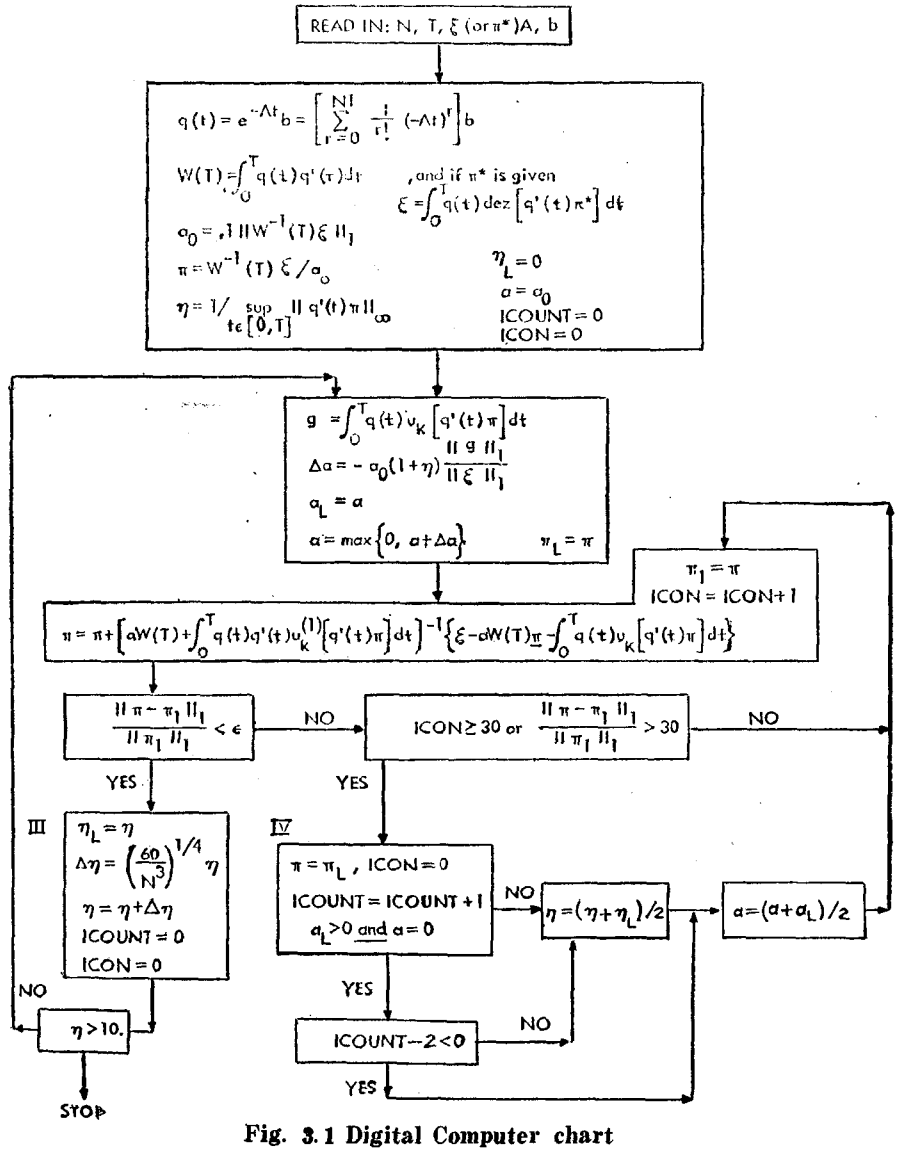


**Fig. 3.1 Digital Computer chart**

the entries be real numbers. This means normal coordinates have been used for clarity, so that the plant matrix shows the eigen values directly.

In this section an iteration of Newton's method will be called just an iteration. A step from one member of the sequence of approximate operator $\{T_k\}$ to the next will be referred to as a step. In plotting the sequence of solution vectors $\{\pi_k\}$ the step number is indicated on the graph. Thus in Fig. 4. 1 the $O$ refers to the vector $\pi_o$ the solution of the linear operator equation $T_o(\pi)=O$;

A summary of the runs made is given in table 4. 1.

**1) Double Integrator Plant**

Two integrators in series form a plant like that of an inertial mass. with control acting on the acceleration, there results.

$$A=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \qquad b=\begin{pmatrix}0\\ 1\end{pmatrix} \tag{4.1}$$

It was chosen for the first set of runs partly because the results can easily be compared with known analytic result.

Run 1 $\xi=\begin{pmatrix}10\\ 2\end{pmatrix}$ $T=15$: $A, b$ of *Eg.* (4.1)

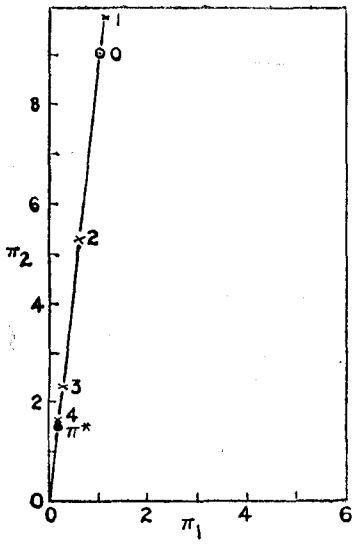


***Fig. 4. 1. Graph of the sequence $\{\pi_k\}$-run 1***

Purpose; To check whether the sequence of operators chosen has the property of sequential convergence. To compare the sequence of solution vector $\{\pi_k\}$ with the solution vector $\pi^*$ of the exact operator. Finally to determine the effect of allowing $\eta$ to become very large.

Resut; See Fig. 4. 1. The sequence of solution vectors $\{\pi_k\}$ appear to lie on a straight line through the region, and they also converge within the numerical accuracy used to the optimal solution vector $\pi^*$. Sequential convergence took place at each step until $\eta$ reached a value of 5,627 at which point the first derivative became too difficult to evaluate.

**Table 4. 1. Computer results**

| Run No. | Name of plant and order | Mode of operation ICHO | M | Convergence of Newton's method |
|---|---|---|---|---|
| 1 | Double | 0 | 15 | Good |
| 2 | integrator | 2 | 120 | Good |
| 3 | Single | 0 | 41 | Fair |
| 4 | oscillator | 0 | 41 | Good |
| 5 | 2 | 0 | 41 | Good |
| 6 | Damped single | 0 | 41 | Good |
| 7 | oscillator 2 | 18 | 41 | Good |
| 8 | Damped double | 0 | 70 | Good |
| 9 | oscillator 4 | 0 | 70 | Good |
| 10 | Double | 2 | 40 | Good |
| 11 | oscillator | 2 | 100 | Good |
| 12 | 4 | 2 | 100 | Fair to good |
| 13 | | 12 | 40 | Good |
| 14 | | 12 | 40 | Good |
| 15 | | 12 | 100 | Good |
| 16 | | | | |
| 17 | | 10 | 25 | Fair |
| 18 | | 12 | 40 | Fair |
| 19 | | 12 | 40 | Poor |

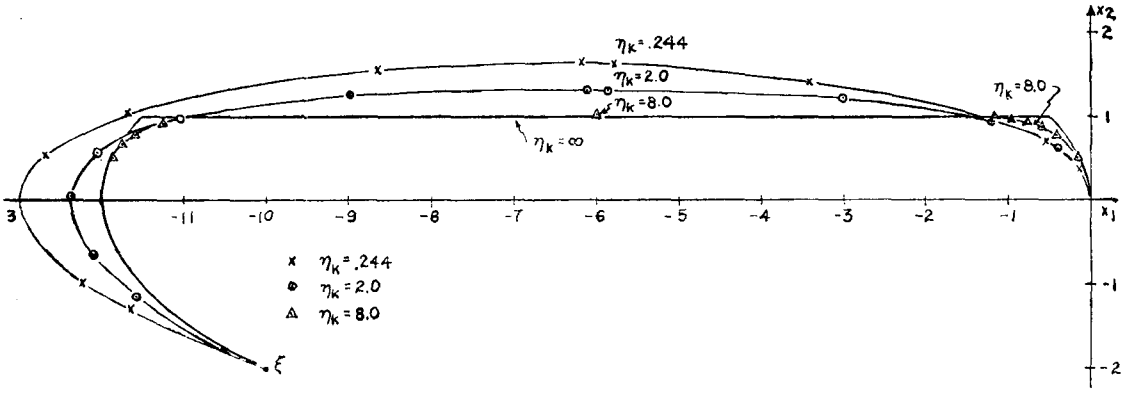


**Fig. 4.2. State space trajectories-run 2**

After the sixth step ($\eta$=8.8) only one iteration of Newton's method was needed per. step. This indicates that the sequence can be carried for beyond the point at which $\pi_k$ converges to $\pi^*$.

Run 2. $\xi=\begin{pmatrix}-10\\-2\end{pmatrix}$ $T$=15: $A$and $b$ of $Eq$: (4.1)

Purpose; To check a symmetric initial condition to Run 1 for symmetry of results. To examine the trajectories in the state space ··· to compare those generated using the approximate operators $T_k(\pi)$ the and the exact operator $T(\pi)$.

Result; The result seems exactly symmetrical to those in Fig. 4.1 to within a very small error. A few chosen state spare trajetories are plotted in Fig. 4.2. It is apparent that as $\eta_k$ increases, the trajectories approach the exact one. From Fig 4.2 one can also conclude that fuel used by the approximate controls converges very closely to the optimal value.

**2) Single Oscillator Plant**

A single degree of freedom oscillator without damping has the system matrix below

$$A=\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} \qquad b=\begin{pmatrix}0\\1\end{pmatrix} \qquad (4.2)$$

This plant was chosen first of all because many physical problems can be modelled by the spring and mass system. Secondly, because it prepares for later work with a two degree of freedom oscillator. Finally this leads to variety: first a plant was used with poles at the origin and now one with poles on the imaginary axis. Runs 3 and 4 were terminated as soon as $\eta_k>2$

Run 3 $\xi=\begin{pmatrix}-2\\2\end{pmatrix}$ $T=\frac{3\pi}{2}$ $A, b$ of $Eq.$ (4.2)

Purpose; To try out a different plant.

Result; The minimum time solution requires a time of $T^*=2\pi-2\ Tan^{-1}\ 2\simeq\frac{3}{2}\pi-0.643$ so this is also a difficult problem. At most of the steps, two attempts were needed to define the next operator $T_k(\pi)$

Run 4 $\xi=\begin{pmatrix}3\\3\end{pmatrix}$ $T=3\pi$ $A$, $b$ of $Eq.$ (4.2)

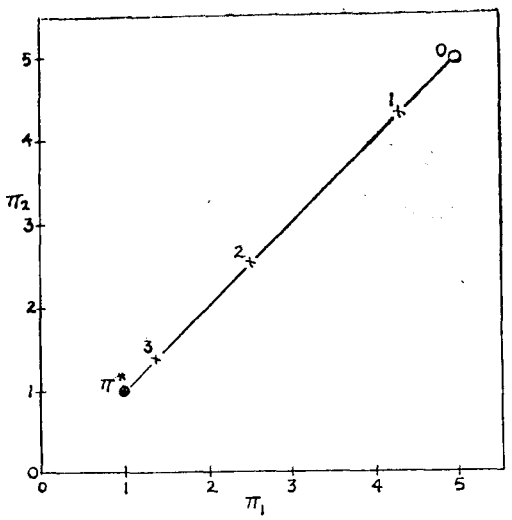


**Fig. 4.3. Graph of sequence $\{\pi_k\}$-run 4**

Purpose; To try another initial condition.

Rusult; The minimum time solution requires a time of $T^*=3\pi-\frac{\pi}{2}-Sin^{-1}\frac{1}{\sqrt{13}}-Tan^{-1}\frac{3}{2}\approx 3\pi-2.835$ only once was it necessary to redefine an operator $T_k(\pi)$ and then needed five iterations for convergence.

The sequence of solution vectors $\{\pi_k\}$ plotted Fig. 4.3 still lies on a stright line.

### 3) Damped Oscillator Plant

In this section tests are described on four different plants. Two are single oscillator and two are double oscillators, all with real damping.

The overall purpose is to try some test on plant whose roots have negative real parts.

A secondary purpose is to try a higher order plant. For plotting purpose the vector $\pi_k$ is split into two vectors of two element each

$$\pi_{1,2}=\begin{pmatrix}\pi_1\\ \pi_2\end{pmatrix} \qquad \pi_{3,4}=\begin{pmatrix}\pi_3\\ \pi_4\end{pmatrix}$$

Then $\pi_2$ is plotted against $\pi_1$, and as separate graph $\pi_4$ is plotted against $\pi_3$.

Run 6
$$A=\begin{pmatrix}-1 & 1\\ -1 & -1\end{pmatrix} \qquad b=\begin{pmatrix}0\\ 1\end{pmatrix}$$
$$\xi=\begin{pmatrix}-2\\ 2\end{pmatrix} \qquad T=\frac{3}{2}\pi$$

Purpose; To test the method with damping present.

Result; One redefining of an operator was required. The total number of iterations needed was 21. The vectors $\pi_k$ no longer lie on a stright line. They are alightly off.

Run 7
$$A=\begin{pmatrix}-.1101 & 1\\ -1 & -.1101\end{pmatrix} \qquad b=\begin{pmatrix}0\\ 1\end{pmatrix}$$
$$\xi=\begin{pmatrix}.5\\ .5\end{pmatrix} \qquad T=2\pi$$

Purpose; To try another initial condition. To compare the fuel costs.

Result; The trajectories in the state space were found to be too close together to be worth plotting for comparison. The sequence of vector $\{\pi_k\}$ appears to lie nearly on a straight line in Fig. 4.4.

Run 8
$$A=\begin{pmatrix}-.1101 & 1 & 0 & 0\\ -1 & -.1101 & 0 & 0\\ 0 & 0 & -.1101 & 2\\ 0 & 0 & -2 & -.1101\end{pmatrix}$$
$$b=\begin{pmatrix}0\\ 1\\ 0\\ 1\end{pmatrix} \qquad \xi=\begin{pmatrix}1\\ 1\\ 3\\ 3\end{pmatrix} \qquad T=2\pi$$

Purpose; To try out the program on a plant with a four dimensional state space.

Result; See Fig. 4.5. The vector $\pi_k$ definitely do not lie on straight line.

Run 9
$$A=\begin{pmatrix}-.5 & 5 & & \\ -5 & -.5 & 0 & \\ & & -.6 & 10.\\ & 0 & -10 & -.6\end{pmatrix}$$
$$b=\begin{pmatrix}0\\ 5\\ 0\\ 6\end{pmatrix} \qquad \xi=\begin{pmatrix}10\\ 10\\ 10\\ 10\end{pmatrix} \qquad T=8$$

Purpose; To try a problem for which the state of the system has many oscillations.

Result; Four iterations were needed at any one step by the time $\eta_k\approx 1$, the vector $\pi_k$ had become very close to its final value.

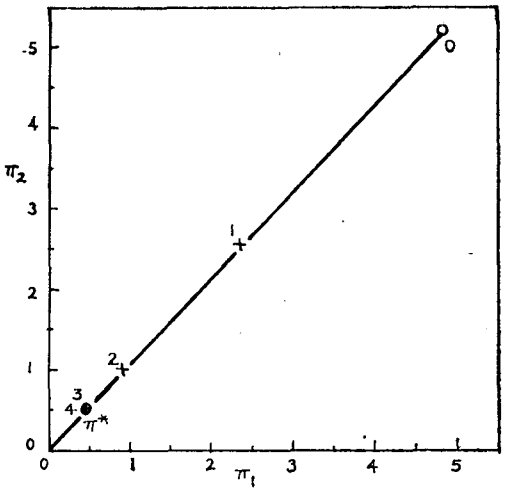


**Fig. 4.4. Graph of sequence $\{\pi_k\}$-run 7**

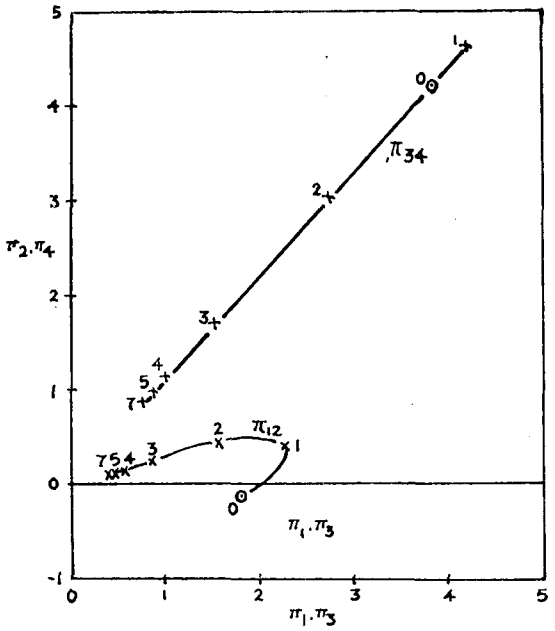


**Fig. 4.5. Graph of the sequence $\{\pi_k\}$-run 8**

**4) Double Oscillator Plant**

This plant is characterized by the matrix

$$A=\begin{pmatrix} 0 & 1 & 0 & \\ -1 & 0 & & \\ & & 0 & w \\ & & -w & 0 \end{pmatrix} \quad b=\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \qquad (4.3)$$

It can be described either as two single degree of freedom oscillators having a common control or as a single oscillator with two degree of freedom. Note that the frequency $w$ is left as a parameter.

At this point the basic features of the program have all been tested. Now questions of accurary and some features of the problems themselves will be examined.

There are four series of runs using this plant. Each series will be described separatly. A vector $l$ is defined for use in these runs

$$l=\begin{pmatrix} \sqrt{3} \\ 1 \\ 1 \\ \sqrt{3} \end{pmatrix}$$

1. Effect of varying $w$-Runs 10-12

In this series the frequency w is varied. As $w \to 1$, the two oscillators become increasingly alike and therefore more difficult to handle with one control.

$$\xi=2l \qquad T=4\pi \qquad (4.4)$$

The purpose is to examine the changes in the sequence of approximate operator as the

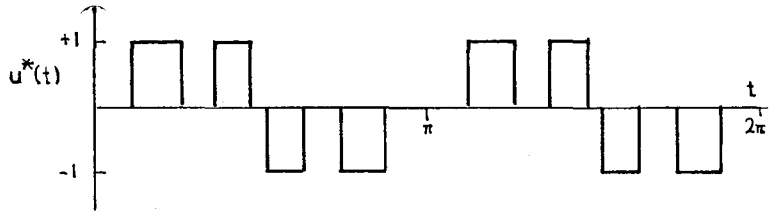


**Fig. 4.6. Graph of fuel optimal control *vs.* time run 10**

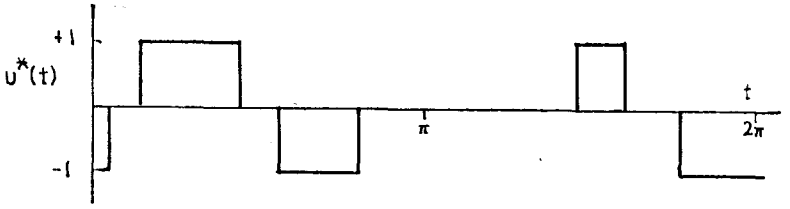


**Fig. 4.7. Graph of fuel optimal control *vs.* time run 11**

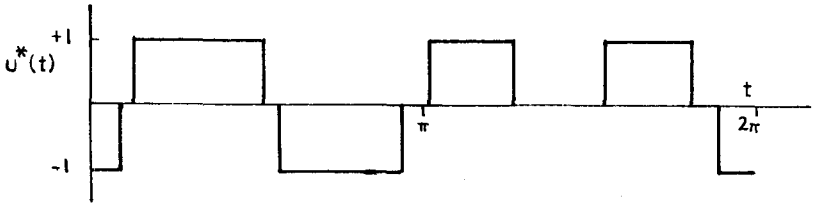


**Fig. 4.8. Graph of fuel optimol contral *vs.* time run 12**

problem becomes more difficult.

A plot of the optimal control variable time was also made to illustrate how the nature of the control changes as the problem become more difficult.

**Table 4.2**

| | | | |
|---|---|---|---|
| Run 10 | W=4 | M=40 | Eqs. 4.3 and 4.4 |
| Run 11 | W=1.5 | M=100 | 〃 |
| Run 12 | W=1.3 | M=100 | 〃 |

In Fig. 4.6. and Fig. 4.7. the total fuel used is very nearly the same. As $w$ is farther decreased to 1.3. in Fig. 4.8. the total fuel used rises sharply.

2. Effect of varying $||\pi^*||$ Run 13—16

These runs were made to explore the relationship between the initial state vector $\xi$ and the optimal costate initial condition vector $\pi^*$. In addition the routine for computing $\xi$ when given $\pi^*$ was checked.

$$W=4. \qquad T=4\pi \tag{4.5}$$

**Table 4.3**

| Run | $\pi^*$ | M | Eqs | $\|\|\xi\|\|_2$ |
|---|---|---|---|---|
| 13 | 5 | 40 | 4.3 and 4.5 | 5.16 |
| 14 | | 40 | 〃 | 6.50 |
| 15 | 2 | 100 | 〃 | 6.91 |
| 16 | 4 | 100 | 〃 | 7.21 |

M: Mode of operation

$||\xi||$: The length of computed vector

The sequence of vector $\{\pi_k\}$ converges to the true vector $\pi^*$ to within the numerical accuracy used as shown in Fig. 4.9. conclusions: All of these runs are in the easy category. The slope $\alpha$ were reduced to zero in one step and none of the operators had to be redefined.

A plot of the initial condition vector $\xi$ is shown in Fig. 4.10. It is apparent that the

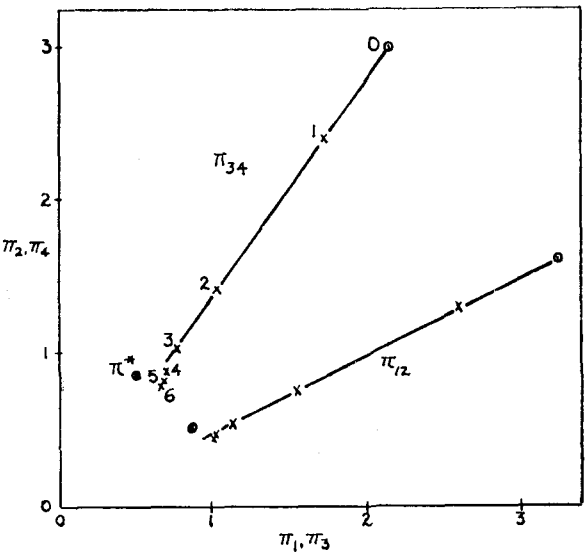


**Fig. 4.9. Graph of the sequence $\{\pi_k\}$-run 13**

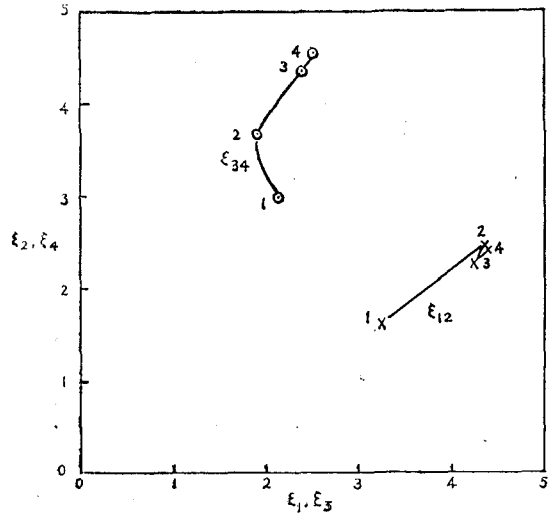


**Fig. 4.10. Computed initial condition vectors $\xi$ runs 13-16**

repected doubling of $||\pi^*||$ leads to diminishing increase in $||\xi||$ is the available control effect become used up.

A plot of the fuel used versus $log_{10}\eta$, for these runs is shown in Fig. 4.11. Notice how well the causes converge to the optimum values.

3. Effect of Decreased Accuracy-Runs 17

A check was made of the effect of decreased accuracy on the procedure. This is done by decreasing $M$, which makes the integration

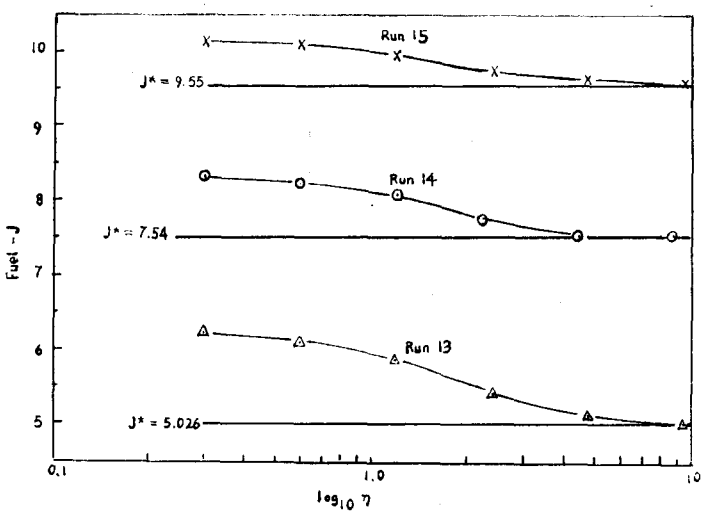


**Fig. 4.11. Graphs of total fuel used *vs log.* $\eta_k$ for the double oscillator plant-$\pi^*$ varied runs 13-15**

step size larger

$W=1.5 \quad T=2\pi \quad \pi^*=2l \qquad (4.6)$

Run 17; $M=25$ *Eqs* (4.3) and (4.6)

Result; The sequence of vectors $\{\pi_k\}$ is shown in Fig. 4.12. The sequence converges must closey to $\pi^*$ when $M=100$.

4. Effect of Nonunique $\pi^*$ Run 18-19

As noted in Theoretical Analysis, theorm I, if the vectors $\varphi(t_i)$, $i=1.2. \cdots\cdots, m.$ do not span the space Rn, then the costate initial condition vector $\pi^*$ is not uniquely specified. With this plant, $T>\pi$ insures uniquences, while $T\leq\pi/2$ leads to less than $n$ switching, andhence to a nonunique $\pi^*$. More precisely, two switchings always occurred, leving two degrees of freedom open. Another test of this type was carried out on the quadrupole plant.

$T=\pi/2 \quad M=40 \quad \pi^*=.5l \qquad (4.7)$

In run 18. The sequence of vector $\{\pi_k\}$ still lies on a straight line. Newton's method seems

**Table 4.4**

| Run | W | Eqs |
|---|---|---|
| 18 | 2 | 4.3 and 4.7 |
| 19 | 1.5 | 〃 |

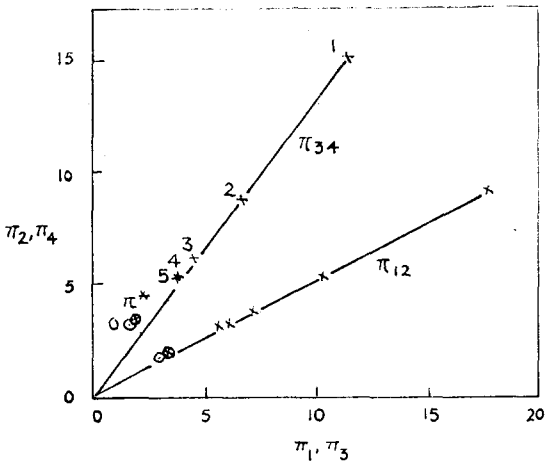


**Fig. 4.12. Graph of the sequence $\{\pi_k\}$ run 17**

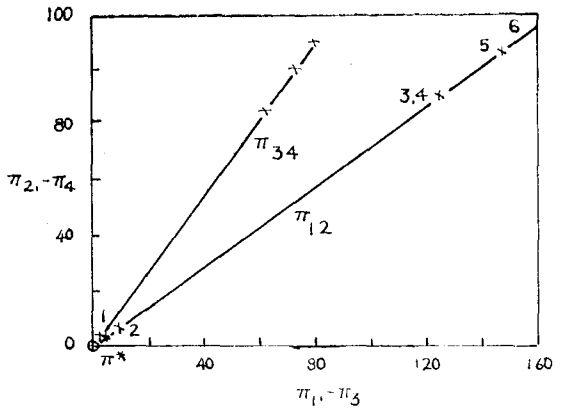


**Fig. 4.13. Graph of the sequence $\{\pi_k\}$ run 18**

to have had no special difficulty in converging.

In run 19. The sequence of solution vectors $\{\pi_k\}$ is shown in Fig. 4. 13. Generally this run was simillar to Run 18. Fig. 4. 14. shows that the sequence of approximate controls still converges, even though $||\pi_k||_2$ is increasing without any apparent bound. The conculsion is that the method seems to work all right.

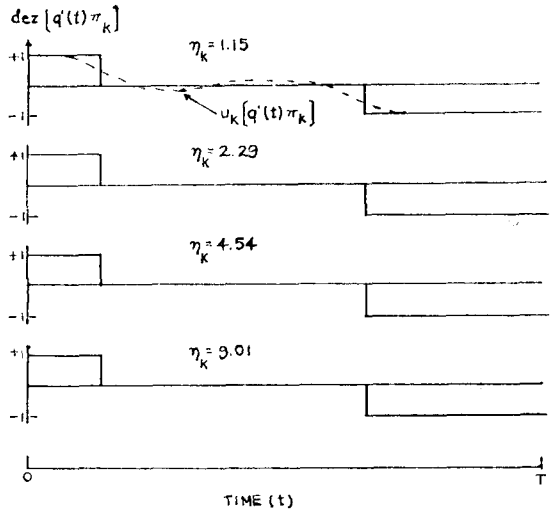


**Fig. 4. 14. Graph of fuel optimal control *vs.* time-run 19**

## 5. Conclusion

For almost all of the problem tried, a convergent sequence of approximations to optimal control was produced. (see Table 4. 1)

In those few cases where the slope $\alpha$ was not reduced to zero (and there the sequence of vectors $\{\pi_k\}$ did not converge to a solution of the necessary conditions) it is suspected that no solution exists.

One of the strength of the method is its flexibility. Thus when Newton's method applied to a particular operator does not converge, another operator is defined until one is found for which convergence does result.

The total fuel used was plotted against $log_{10}\bar{\eta}_k$, since the approximate control function $u_k(\cdot)$ is an exponential type of function. When the parameter $\eta_k$ reaches a value of $\eta=5$, usually the resualting cost is within 1% of $J^*$, the optimal fuel cost. Note that the first operator usually has a cost 5 to 30% greater than $J^*$, showing in practice the efficiency of the approximate controls $u_k$.

About effectiveness of the approximate operator: The sequence of solution vecter $\{\pi_k\}$ generally had moved close to its final value when the parmeter $\eta_k$ had reached a value of $=2\sim5$. Then the accuracy used in the digital computations becomes a increasingly important factor in determing the distance between $\pi_k$. and $\pi^*$. As $\eta_k$ increase beyond $\eta=10$, usually only one iteration is enough to meet the criterion for convergence of Newton's method.

About accuracy: In Run 10, 11 and 17 a study was made of the effect of interval size or number of subdivisions in each integration. Note that no matter how poor the integration scheme used: (1) the state space still gets to the origin at time $t+T$ for model used, and (2) the run can always be repeated except for round off error in the digital computer.

As accuracy was decreased, the number of iterations of Newton's method required increased.

About straight line behavior of $\{\pi_k\}$: When the system (open loop) poles lie on the imaginary axis, the sequence of vector $\{\pi_k\}$ was found to lie on a straight line through the origin. This is true also for the fourth order and sixth order examples in the vector spaces $R_4$ and $R_6$, respectively. No reason has been found for this. It is surmised that the straight line behavior will hold for any sequence of control approximations $u_k$ having symmetry about the origin, acting on a conservative system in a fixed time control problem.

If the control were linear with slop $\alpha$, the costate initial conditions would be

$$\pi=1/2w^{-1}(T)\ [\xi-e^{-AT}\theta]$$
$$=1/2w^{-1}(T)\xi$$

The key of this of this design is to use the costate resulting from above mentioned form with the optimal control function - $dez(\cdot)$. Because of the straight line behavior of $\{\pi_k\}$ there is some slop $\alpha$ for which this gives the optimal result. The effect of $\alpha$ is to determine $||\pi||_2$, the lenth of the vector $\pi$.

we set

$$u(t) = -dez[1/2q'(t)w^{-1}(T)\xi]$$

It remains to choose the constant $\alpha$, clearly, the smaller $\alpha$ is the closer the control to a time optimal one; the larger $\alpha$ is the slower the control to a time optimal one; the larger $\alpha$ is the slower the control but also the more efficient in its use of fuel. The safest way to pick $\alpha$ is by test of the system under field conditions.

With certain system it may be possible to design a rule for choosing. For instance, in the single oscillator control problem to the origin, the time optimal control reduces $||x(t)||_2$ by about two units every $\pi/w$ seconds, so the minimum time $T^*$ is approximately

$$T^* \approx \frac{\pi}{2w}||\xi||_2$$

One way to choose $\alpha$ would be to give the control argument a magnitude based on the ratio $T^*/T$. For example, let the magnitude be $1.0+T^*/T$. Then as $T^*/T \to 0$ the control effort also goes to zero. For clarity this will be done in two steps. First, change the magnitude of the argument to one

$$\frac{q'(t)w^{-1}(T)\xi}{|q'(O)w^{-1}(T)\xi|}$$

Then multiplying by the chosen magnitude yields the desired control.

$$u(t) = -dez\left\{\left(1+\frac{\pi}{2wT}||\xi||_2\right)\frac{q'(t)w^{-1}(T)\xi}{|q'(O)w^{-1}(T)\xi|}\right\}$$

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