On the Critical Mass of the Superdense Star

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高密度星의 限界質量에 關하여

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論 文 概 要

恒星進化의 最終段階에서 이루어지는 高密度星(白色矮星, 中性子星)의 平衡의 限界質量은 現在까지 太陽質量의 0.6~2倍로 알려졌다. 이 값의 差異는 假定한 狀態方程式에 따르고 있으나 그 어느 것이나 모두 太陽의 質量과 大差없다는 事實은 注目할 일이다. 또 觀測된 普通의 별의 質量도 太陽質量의 $\frac{1}{10}$~60倍의 범위에 있다. 이두 事實을 綜合하여 太陽의 質量이 가지는 特定한 자리의 物理的인 根據로서 Polytrope 氣體의 指數 $(\gamma=\frac{4}{3})$와 $\left(\frac{\hbar c}{Gm_H{}^2}\right)^{\frac{3}{2}}$의 指數사이의 聯關性을 考察하였다.

1. Introduction

Recent calculations of the critical masses of superdense stars(white dwarfs, neutron stars and hyperon stars) have shown that they are about one solar mass irrespective of their assumed equations of state. This remarkable fact has not been explained in terms of the theoretical grounds. In the present investigation, a rather simple explanation has been attempted. Since one solar mass is also a typical value for the ordinary stars, the latter being in the range of one tenth up to 60 solar masses, we may say that the ordinary star has a loosely defined "critical mass" of around 1 solar mass. We may make this statement somewhat plausible by the following argument in section 2. Then the existence of the more sharply defined critical mass in case of superdense stars is investigated in section 3.

2. Mass of ordinary stars

We shall start with the equation of gravitational equilibrium for the star,

$$\frac{dP}{dr}=-\frac{G\rho m}{r^2} \qquad (1)$$

where P, ρ, m are the pressure, density and mass as functions of r, distance from the center, and G the gravitation constant $=6.67\times10^{-8}$ c.g.s.

By ordinary stars, we mean the stars that have central temperature of the order of $10^7\sim10^8$ °K at the densities of $0.1\sim100\,g/cm^3$. As was pointed out by Zeldovich (Zeldovich and Novikov, 1971) these stars are the "hot stars" to the contrast to the "cold stars" which will be considered in section 3.

In the overall temperature and the density ranges of the "ordinary stars", we may assume the equation of state for the ideal gas with the radiation pressure of photons as a necessary correction to the gaseous pressure of the plasma. The ratio of contribution to the total pressure by the photon to that by the plasma, however, is rather critical in determining the mass of the ordinary stars from the point of view of "the cloud-bound physicist", the classical invention of Eddington (Eddington 1926).

The total pressure P is composed of two parts, P_R due to the radiation and P_G due to the matter

$$P=P_R+P_G=\frac{1}{3}aT^4+\frac{k}{\mu m_H}\rho T. \quad (2)$$

where a is the black body constant= $\pi^2k^4/15(c\hbar)^3=7.56\times10^{-15}$ c.g.s., k is Boltzmann constant $(=1.38\times10^{-16}$c.g.s.), μ is the mean molecular weight, and m_H is the mass of hydrogen atom $(=1.67\times10^{-24}$g).

If we denote the ratio P_G/P by β, then the ratio of P_R to P_G is given by

$$\frac{P_R}{P_G}=\frac{1-\beta}{\beta}=\frac{\mu m_H a}{3k}(T/\rho^{\frac{1}{3}})^3. \quad (3)$$

Hence the ratio is proportional to the cube of $T/\rho^{1/3}$, while the stellar mass M also depends on this parameter, as is shown below (Zeldovich and Novikov 1971).

From the dimensional consideration of eq. (1), it is clear that

$$\frac{\langle P\rangle}{R}=\langle\rho\rangle\frac{GM}{R^2}, \quad (4)$$

where $\langle\ \rangle$ denotes the appropriate average taken over the whole star and M and R are the total mass and the radius of the star. Since $\rho\sim\frac{M}{R^3}$, (4) is written as

$$P=k_1\ GM^{\frac{2}{3}}\rho^{4/3}, \quad (5)$$

where k_1 is a constant ($\sim$1) depending on the method of taking average. From (3) and (5), we have

$$M=\frac{\sqrt{45}}{k^{3/2}\hbar}\left[\left(\frac{\hbar c}{Gm_H^2}\right)^{\frac{3}{2}}m_H\frac{1}{\mu^2}\frac{(1-\beta)^{\frac{1}{2}}}{\beta^2}\right]$$

$$\sim100\frac{(1-\beta)^{\frac{1}{2}}}{\mu^2\beta^2}\text{(solar mass)}, \quad (6)$$

where we may note the appearance of a factor $\left(\frac{\hbar c}{Gm_H^2}\right)^{\frac{3}{2}}m_H$ in this mass formula. As $\mu\sim1$, the mass is mainly determined by this factor for given value of β.

Eq. (6), however, does not give any meaningful range of mass, because the factor $\frac{(1-\beta)^{\frac{1}{2}}}{\beta^2}$ varies from 0 to infinity as β changes from 1 to 0.

From eq. (5), it is immediately clear that in case of $P\propto\rho^{\frac{4}{3}}$ the mass M is uniquely determined for the group if the stars having the same proportionality constant. This is the case of the polytropic star $(P\propto\rho^\gamma)$ with $\gamma=\frac{4}{3}$ or the polytropic index $n=3$. In this case, the proportionality constant is relalated to the entropy per particle. If the the chemical composition (i.e.μ) is the same then the total entropy S remains the same for these stars.

In other words, the stars of the same μ and S have the same mass if $P\propto\rho^{\frac{4}{3}}$. It follows that $P\propto\rho^{\frac{4}{3}}$ is an important case of determining some range of the stellar mass if the range of the entropy is known.

We will see the case of "cold" stars as a particular one where $S=0$ (the range is 0, too), which will be considered in the next section.

3. The superdense star

By the superdense star or cold star, we mean the star at the end of its evolution having exhausted its nuclear fuel and at extremely high density after the continued contraction. Their density ranges from 10^4—10^{11} g/cm^3 (white dwarf) to 10^{11}—10^{14} g/cm^3 (neutron star), and their states are independent of T unless $T > 10^8$, so we may use zero temperature approximation of the degenerate matter.

In case of the cold star, the main contribution to the pressure and the energy comes from the degenerate gas of the electron for the white dwarf and of the neutron for the neutron star respectively.

The degenerate electron pressure P_e and the density ρ are given by (Chandrasekhar 1939)

$$P_e = \frac{\pi m_e^4 c^5}{3h^3} f(x) \qquad (7)$$

$$\rho = \frac{8\pi m_N m_e^3 c^3}{3h^3} x^3. \qquad (8)$$

where m_e is the electron mass, h is Planck's constant (6.625×10^{-27} c.g.s.), $x = \frac{p_F}{m_e c}$, p_F is Fermi momentum, and $f(x)$ is given as

$$f(x) = x(2x^2-3)(x^2+1)^{\frac{1}{2}} + 3\sinh^{-1}x. \qquad (9)$$

which goes assymptotically to

$$\begin{cases} \frac{3}{5}x^5 \text{ when } x \ll 1 \text{ (non-relativistic case)}, & (10) \\ 2x^4 \text{ when } x \gg 1 \text{ (relativistic case)}. & (11) \end{cases}$$

For the neutron star, the electronic mass m_e should be replaced by the neutron mass m_N.

Hence from (7) (8) (10) (11), we have as limitting cases

(a) non-relativistic

$$P \propto \rho^{\frac{5}{3}} \qquad (12)$$

(b) relativistic

$$P \propto \rho^{\frac{4}{3}}, \qquad (13)$$

which can be alternatively expressed as $P \propto \rho^\gamma$, the well known polytropic relation with the polytropic index $n = \frac{1}{\gamma-1} = \frac{3}{2}$ and 3.

For the "cold" star, we take the case (b), i.e. the relativistic one. Then, we have from (7), (8), and (11)

$$P_e = \frac{\hbar c}{12\pi^2}\left(\frac{3\pi^2}{m_N \mu}\right)^{\frac{4}{3}} \rho^{\frac{4}{3}}. \qquad (14)$$

Substitution of (14) into (5) yields the Chandrasekhar limit as

$$M = \frac{1}{2}(3\pi)^{\frac{1}{2}}(2.02)\frac{1}{\mu^2}\left(\frac{\hbar c}{G m_N^2}\right)^{\frac{3}{2}} m_N$$
$$= 5.87 \frac{1}{\mu^2} \text{(solar mass)}, \qquad (15)$$

For the neutron star, the similar calculation leads to ($m_e \to m_N$)

$$M = 0.7 \text{ (solar mass)}, \qquad (16)$$

which is the Oppenheimer-Volkoff limit.

At the super-nuclear density $\rho > 10^{14}$ g/cm^3, the uncertainty in the knowledge of the equation of state due to diversity of particle interactions prevents us from obtaining the detailed study of superdense matter. Ambartsumian and Saakyan (1969) found that there appear, besides neutrons and a small number of protons and electrons, many other types of elementary particles e.g. muons, pions, hyperons. But theoretical study (Harrison, Thorne, Wakano and Wheeler 1965) has shown the existence of denumerable infinity of critical masses that get smaller as the central density tends to infinity provided the equation of state follows γ-law (i.e. $P \propto \rho^\gamma$) asymptotically.

This means that even if the central density of the "cold" star approaches infinity the critical mass remains around 1 solar mass!

4. The significance of one solar mass

As we have previously noted, the appearance of $\left(\frac{\hbar c}{Gm_H^2}\right)^{\frac{3}{2}}$ in the mass formula (6) seemes to be responsible for producing the right order of magnitude of a solar mass (2×10^{33}g). The dimensionless number, $\frac{\hbar c}{Gm_H^2}$, in the parenthesis has been argued frequently in various literatures in connection with the attempt to find any link between the cosmical and the subatomic phenomena, for it contains the fundamental constants, G, m_H, h, and c. The number $\frac{\hbar c}{Gm_H^2}=1.691\times10^{38}$ involved in our stellar problem is contrasted with the analogous number (the fine structure constant) $\frac{\hbar c}{e^2}=137.037$ in electromagnetic interactions of the subatomic physics. Wheeler has pointed out (Wheeler 1964) that "the enormity of this number is testimony to the many nucleons that have to be present before gravitational interactions overwhelm all other forces".

A question remains, however, "Why a solar mass?" in view of its rather ubiquitous appearance as the critical mass in cases we have seen. One solar mass takes some $\left(\frac{\hbar c}{Gm_H^2}\right)^{\frac{3}{2}}$ times of m_H, or is equal to $\frac{\hbar^{\frac{3}{2}}\ c^{\frac{3}{2}}}{G^{\frac{3}{2}}m_H^2}$. On the other hand, the simplest expression of a mass containing G, h, m_H and c is known to be $\left(\frac{\hbar c}{Gm_H^2}\right)^{\frac{1}{2}}$ times m_H, or equal to $\left(\frac{\hbar c}{G}\right)^{\frac{1}{2}}$ which contains actually no m_H at all. This hypothetical mass has been unpopular, for it was too large compared to the nucleonic mass m_N. No particle of importance with this mass has been found so far, although some authors believe in favor of its fundamental significance (Zeldovich and Novikov 1971).

Now what have we got to say about our "one solar mass"? Do we have anything better in favor of "our sun" than of the hypothetical mass? Since we are considering the link between the long-range (i.e. gravitation) and the short-range force (i.e. radiation or degenerate pressure) we need to re-examine these 4 constants of their roles in their inter-relationship.

G and m_H go together with the gravitational force in the equation of equilibrium (1). The constants c and $\hbar$ stem from the radiation law in case of the ordinary stars, but with the cold star they come from the relativistic expression of energy or momentum in the formula giving the degenerate gas pressure. While the radiation law applies in any range of T, the law of degenerate gas pressure is almost independent of T, or only exact at $T=0$. This fact seems to be the cause of the distinction in the range of mass of ordinary stars and superdense stars.

While the instability in the ordinary star of excessive mass is apt to cause the explosion, the superdense star beyond the critical mass is known to collapse gravitationally without any choice of alternative equilibrium configuration (Harrison, Thorne, Wakano and Wheeler, 1965).

As to the reason for the number $\left(\frac{\hbar c}{Gm_H^2}\right)^{\frac{3}{2}}$ times m_H giving the right order of magnitude of a solar mass, we may ask "Why is $\left(\frac{\hbar c}{Gm_H^2}\right)$ raised to $\frac{3}{2}$ power and not to any other value?" For instance $\frac{1}{2}$ gives the hypothetical mass $\left(\frac{\hbar c}{Gm_H^2}\right)^{\frac{1}{2}} m_H=\left(\frac{\hbar c}{G}\right)^{\frac{1}{2}}=2\times10^{-4}$g which is too large as compared to that of an elementary particle, and too small to be actual solar mass. Its second power, $\left(\frac{\hbar c}{Gm_H^2}\right)^2$ times m_H, gives $\sim10^{20}$ solar mass which is presumably as large as the mass of the universe !

If we recall the argument at the end of section 2, the answer might be as follows:

From the derivation of eq.(6), the exponent $\frac{3}{2}$ is traceable in eq.(5) where M is raised to $\frac{2}{3}$ power with G to the first although it seems at first quite arbitrary to make it $\frac{2}{3}$. But the sum of the exponents of M and ρ should be 2 from the dimensional requirement of the eq.(5). Hence the condition of polytropic relation $P\propto\rho^\gamma$ with $\gamma=\frac{4}{3}$ ($n=3$) need to be changed to other value of γ in order to get other exponent for $\left(\frac{\hbar c}{Gm_H^2}\right)$. In other words, the necessity of $\frac{3}{2}$ power of $\left(\frac{\hbar c}{Gm_H^2}\right)$ stemmed from γ-value of the polytropic relation. In this context, it might be worth of noting the fact that the pure radiation behaves as a gas of polytropic gas of $\gamma=\frac{4}{3}$.*

So we have to change the law of radiation to get other power for the number $\left(\frac{\hbar c}{Gm_H^2}\right)$. We may, as a matter of fact, trace the origin of the exponent $\frac{3}{2}$ to the same law.

5. Discussion

We have so far, at least, made it plausible that the critical mass of the superdense star and the observed mass of ordinary star hover around one solar mass with some physical reason. In view of complexity and diversity in the physical structure of the star, the result should be far from being called any proof or the like. It is easily seen that the limiting mass M_1 ($\rho_c\to\infty$), should be about one solar mass, in case of assymptotic γ-law for the equation of state with $\gamma=\frac{4}{3}$. For other values of γ, our simplified argument does not apply and we have to treat the problem in full detail instead of using the averaged physical quantities, P,ρ,T etc. in eq. (4), (5), and (6). Since we have $1<\gamma<2$, the lower limit being required against the dynamical instability, the upper by the causality (i.e. the speed of sound v_s and the speed of light c with the relation $\gamma-1=\left(\frac{v_s}{c}\right)^2$), we may fairly expect the limiting mass should not be drastically different from its value when $\gamma=\frac{4}{3}$, i.e. about one solar mass.

It seems interesting that our sun, besides its coincidental proximity to us, might possess far more fundamental significance in understanding the possible link between the universe and the micro-universe.

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* This is no coincidence since the extremely relativistic particle in the cold star behaves more like photon as $pc\gg mc^2$ in the energy expression $E=(p^2c^2+m^2c^4)^{\frac{1}{2}}$.

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