JACOBI POLYNOMIAL AND POTENTIAL THEORY

By Manilal Shah

1. Introduction. In this paper Jacobi polynomial has been utilized to establish the solutions of fundamental differential equation in potential theory. Methods which we have employed are based upon the application of orthogonality properties for Bessel-functions and known integral. Results derived in this paper are of great importance in the mathematical analysis.

We shall deal with the problem for determining a function $\phi(r,z)$ for the half space $a\geq r\geq 0$, $z\geq 0$, satisfying the differential equation [(3), (36.1), p. 135]:

$$(1.1)\quad \frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}=0,$$

subject to the boundary conditions

(1.2) $\phi\to 0$ as $z\to\infty$,

(1.3) $\dfrac{\partial\phi}{\partial r}+k\phi=0$ on $r=a$,

(1.4) ϕ remains finite as $r\to 0$; and the initial condition

(1.5) $\phi=f(r)$, on $z=0$.

We shall consider the following value of $f(r)$ in the present investigation

$$(1.6)\quad f(r)=r^{2\tau+2\sigma}(a^2-r^2)^{\alpha}P_{\mu}^{(\alpha,\beta)}\left(2\frac{r^2}{a^2}-1\right)$$

where $P_n^{(\alpha,\beta)}(x)$ is a Jacobi polynomial which can be converted to either Gegenbauer, Legendre and Tchebichef polynomials with proper choice of parameters. Any function which is bounded and has a finite number of maxima and minima can be represented by a series of Bessel-functions, hence $f(r)$ of the form (1.6) and is of general character.

2. In our present work, we shall require the following formulae in the proof.

(A) Integral [(4), (2, 2)]:

$$(2.1)\quad \int_0^a r^{2\tau+2\sigma+1}(a^2-r^2)^{\alpha}P_{\mu}^{(\alpha,\beta)}\left(2\frac{r^2}{a^2}-1\right)J_0(\lambda_m r/a)dr$$

$$=\frac{a^{2\tau+2\alpha+2\sigma+2}\Gamma(\alpha+\mu+1)\Gamma(1+\sigma+\tau)\Gamma(1+\sigma+\tau-\beta)}{2\Gamma(1+\sigma+\tau-\beta-\mu)\Gamma(2+\sigma+\tau+\alpha+\mu)\ \mu\,!}$$

$$\times{}_2F_3\left[\begin{matrix}1+\sigma+\tau,1+\sigma+\tau-\beta\\1,1+\sigma+\tau-\beta-\mu,2+\sigma+\tau+\alpha+\mu\end{matrix};-\lambda^2 m/4\right]$$

where Re $(\alpha)>-1$, Re $(\tau+\sigma)>-1$.

(B) Orthogonality properties for Bessel-functions [(1), p. 71(49) and p. 70, (48)]:

$$(2.2)\ \int_0^1 tJ_\gamma(\lambda_m t)J_\gamma(\lambda_n t)dt = \begin{cases} 0, & \text{if } n\neq m, \\ \frac{1}{2}\lambda_m^{-2}\left\{\lambda_m^2[J_\gamma'(\lambda_m)]^2+(\lambda_m^2-\gamma^2)[J_\gamma(\lambda_m)]^2\right\}, & \text{if } n=m, \end{cases}$$

and

$$(2.3)\ \int_0^1 tJ_\gamma(\gamma_m t)J_\gamma(\gamma_n t)dt = \begin{cases} 0, & \text{if } n\neq m \\ \frac{1}{2}[J_{\gamma+1}(\gamma_m)]^2, & \text{if } n=m. \end{cases}$$

3. Solutions of the problem: *The solutions of the problem to be established are:*

First solution:

$$(3.1)\ \phi(r,z)=\frac{a^{2\tau+2\alpha+2\sigma}\ \Gamma(\alpha+\mu+1)\Gamma(1+\sigma+\tau)\Gamma(1+\sigma+\tau-\beta)}{\mu!\Gamma(1+\sigma+\tau-\beta-\mu)\Gamma(2+\tau+\alpha+\sigma+\mu)}$$

$$\times\sum_i {}_2F_3\left(\begin{matrix}1+\sigma+\tau, 1+\sigma+\tau-\beta \\ 1, 1+\sigma+\tau-\beta-\mu, 2+\tau+\alpha+\sigma+\mu\end{matrix}; -a^2w_i^2/4\right)\frac{e^{-zw_i}J_0(rw_i)}{\{J_1(aw_i)\}^2+\{J_0(aw_i)\}^2}$$

where Re $(\alpha)>-1$, Re $(\tau+\sigma)>-1$.

Second Solution:

$$(3.2)\ \phi(r,z)=\frac{a^{2\tau+2\alpha+2\sigma}\Gamma(\alpha+\mu+1)\Gamma(1+\sigma+\tau)\Gamma(1+\sigma+\tau-\beta)}{\mu!\ \Gamma(1+\sigma+\tau-\beta-\mu)\Gamma(2+\tau+\alpha+\sigma+\mu)}$$

$$\times\sum_i\frac{e^{-zw_i}J_0(rw_i)}{[J_1(aw_i)]^2}\ {}_2F_3\left[\begin{matrix}1+\sigma+\tau, 1+\sigma+\tau-\beta \\ 1, 1+\sigma+\tau-\beta-\mu, 2+\sigma+\tau+\alpha+\mu\end{matrix}; -a^2w_i^2/4\right]$$

valid for Re $(\alpha)>-1$, Re $(\tau+\sigma)>-1$.

PROOF. The solution of (1.1) subject to the given conditions can be written in the form [(3), (36.4), p. 136]:

$$(3.3)\quad \phi(r,\ z)=\sum_i N_i\, e^{-zw_i}J_0(w_i r),$$

where the sum is taken over the positive roots of the transcendental equation.

$$(3.4)\quad w_iJ_0'(w_ia)+kJ_0(w_ia)=0.$$

By the virtue of the initial condition (1.5), the coefficients of N_i must be properly chosen to satisfy the following relation.

$$(3.5)\quad f(r)=\sum_i N_iJ_0(rw_i)$$

In view of (1.6), we have

$$(3.6)\ r^{2\tau+2\sigma}(a^2-r^2)^\alpha P_\mu^{(\alpha,\beta)}(2\frac{r^2}{a^2}-1)=\sum_i N_i\, J_0(rw_i)$$

where $a\geq r\geq 0$ and the validity of this expansion is justified since $f(r)$ is continuous and of bounded variation in the open interval $(0, a)$.

Multiply both sides of (3.6) by $rJ_0(rw_j)$ and integrate with respect to r from 0 to a, change the order of integration and summation on the right which we suppose to be permissible due to the absolute convergence of integral and summation involved in the process, we have

$$(3.7)\ \int_0^a r^{2\tau+2\sigma+1}(a^2-r^2)^\alpha P_\mu^{(\alpha,\beta)}\left(2\frac{r^2}{a^2}-1\right)J_0(rw_j)dr$$
$$=\sum_i N_i\int_0^a rJ_0(rw_j)J_0(rw_i)dr.$$

Now making use of orthogonality properties (2.2) and (2.3) with $t=r/a$, $\gamma=0$, $\lambda_m=aw_j$, $\lambda_n=aw_i$ etc., on the right and (2.1) replacing $\lambda_m=aw_j$, on the left of (3.7), we obtain

$$(3.8)\ N_j=\frac{a^{2\tau+2\alpha+2\sigma}\Gamma(\alpha+\mu+1)\Gamma(1+\sigma+\tau)\Gamma(1+\sigma+\tau-\beta)}{\mu!\Gamma(1+\sigma+\tau-\beta-\mu)\Gamma(2+\tau+\alpha+\sigma+\mu)\left[\{J_0'(aw_j)\}^2+\{J_0(aw_j)\}^2\right]}$$
$$\times{}_2F_3\left(\begin{matrix}1+\sigma+\tau, 1+\sigma+\tau-\beta\\ 1, 1+\sigma+\tau-\beta-\mu, 2+\tau+\alpha+\sigma+\mu\end{matrix};\ -a^2w_j^2/4\right),$$
$$\text{Re }(\alpha)>-1,\ \text{Re }(\tau+\sigma)>-1,$$

and

$$(3.9)\ N_j=\frac{a^{2\tau+2\alpha+2\sigma}\Gamma(\alpha+\mu+1)\Gamma(1+\sigma+\tau)\Gamma(1+\sigma+\tau-\beta)}{\mu!\ \Gamma(1+\sigma+\tau-\beta-\mu)\Gamma(2+\tau+\alpha+\sigma+\mu)\left[J_1(aw_j)\right]^2}$$
$$\times{}_2F_3\left(\begin{matrix}1+\sigma+\tau, 1+\sigma+\tau-\beta\\ 1, 1+\sigma+\tau-\beta-\mu, 2+\sigma+\tau+\sigma+\mu\end{matrix};\ -a^2w_j^2/4\right).$$
$$\text{Re }(\alpha)>-1,\ \text{Re }(\tau+\sigma)>-1.$$

The solutions (3.1) and (3.2) obtain immediately in view of (3.3) and (3.8) with the use of known relation $J_0'(z)=-J_1(z)$ and (3.3) and (3.9) respectively.

Particular case: Setting $\alpha=\beta=\gamma-\frac{1}{2}$ and using the known relation [(2), p. 267]:

$$C_n^\gamma(x)=\left((2\gamma)_n/(\gamma+\tfrac{1}{2})_n\right)P_n^{\left(\gamma-\frac{1}{2},\ \gamma-\frac{1}{2}\right)}(x)$$

etc., in (3.1), we obtain a known result given by the author [(5), eqn. (3.1)]:

$$\phi(r,z)=\frac{a^{2\tau+2\gamma+2\sigma-1}\Gamma(\gamma+\frac{1}{2})\Gamma(2\gamma+\mu)\Gamma(1+\sigma+\tau)\Gamma\left(\sigma+\tau-\gamma+\frac{3}{2}\right)}{\mu!\Gamma(2\gamma)\Gamma\left(\frac{3}{2}+\sigma+\tau-\gamma-\mu\right)\Gamma\left(\frac{3}{2}+\sigma+\tau+\mu+\gamma\right)}$$
$$\times\sum_i\frac{e^{-zw_i}J_0(rw_i)}{\left[\{J_1(aw_i)\}^2+\{J_0(aw_i)\}^2\right]}\ {}_2F_3\left(\begin{matrix}1+\sigma+\tau,\ \frac{3}{2}+\sigma+\tau-\gamma\\ 1,\ \frac{3}{2}+\sigma+\tau-\gamma-\mu,\ \frac{3}{2}+\sigma+\tau+\mu+\gamma\end{matrix};-\frac{a^2w_i^2}{4}\right)$$

where Re $(\gamma) > -\frac{1}{2}$, Re $(\tau+\sigma) > -1$.

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REFERENCES

[1] A. Erde'lyi, ***Higher transcendental functions***, Vol. II, McGraw-Hill Book Company, New York, 1953.

[2] A. Erde'lyi, ***Tables of integral transforms***, Vol. II, McGraw-Hill Book Company, New York, 1954.

[3] I.N. Sneddon, ***Special functions of mathematical physics and chemistry***, Oliver and Boyd., 1961.

[4] M. Shah, ***Jacobi polynomials associated with cooling of a heated cylinder***, Proc. Camb. Phil. Soc. (Under-communication).

[5] M. Shah, *Use of Gegenbauer (Ultraspherical) polynomials in potential theory*, Analele Universitatii Din Timisoara, Seria Matematica, Vol. VII, fasc. 2, 1969, Timisoara, Romania.