Bull. Korean Math. Soc.
Vol. 8 (1971), pp. 17-19

On topological N-groups

Jaihan Yoon

1. By V. S. Carin, 1962, a question raised, essentially as follows: is a topological N-group an N-group in abstract sence? Meanwhile, a counter example that disproves the qustion is given by Platonov [1], and also it is shown that not every closed subgroup of a topological N-group is such a one[2].

In contrast with topological N-groups, we shall see in the present note, that every closed subgroup of a topological $\dot{N}$-group is itself a topological $\dot{N}$-group, and also that a quotient group of a locally compact topological $\dot{N}$-group over a maximal closed subgroup is an abstract $\dot{N}$-group.

Let us recall that a topological group is a topological N-group if it satisfies the normalizer condition for closed subgroups, and that the normalizer of a closed subgroup is a closed subgroup[1].

A topological $\dot{N}$-group will be defined below after the fashion of an abstract $\dot{N}$-group, except for topological notion "closed", so that an N-group is an $\dot{N}$-group in the topological sense.

2. Let us start by giving the definition of topological $\dot{N}$-groups.

DEFINITION. A topological group G is a topological $\dot{N}$-group if for any closed subgroup H, each maximal subgroup K that is closed in H is normal in H.

PROPOSITION 1. *If G is a topological N-grop, it is a topological $\dot{N}$-group.*

Proof. Let H be any closed subgroup of G, and K a closed maximal subgroup of H. We may assume, without the loss of the generality, that K is properly contained in H. Then the normalizer $N(K)$ of K is closed [3] and distinct from K, and hence containes H. This proves that K is normal in H.

It is clear that every closed or open subgroup of a topological $\dot{N}$-group is itself an $\dot{N}$-group.

LEMMA. *The quotient group G/H of a topological $\dot{N}$-group is also a topological $\dot{N}$-group, where H is a closed normal subgroup of G.*

Proof. Let A be a closed subgroup of a quotient group G/H and B a closed maximal subgroup of A. The continuity of the natural projection φ implies that $\varphi^{-1}(B)$ and $\varphi^{-1}(A)$ are closed in G.

We shall show that for any x in $\varphi^{-1}(A)\setminus\varphi^{-1}(B)$, the subgroup $\{x, \varphi^{-1}(B)\}$ generated by x and $\varphi^{-}(B)$ is $\varphi^{-1}(A)$, namely $\varphi^{-1}(B)$ is maximal in $\varphi^{-1}(A)$. Since B is maximal in A, the subgroup generated by $\varphi(x)$ and B coincides with A, that is $\{\varphi(x), B\}=A$. Hence, for each a in $\varphi^{-1}(A)$, we have

$$\varphi(a)=\varphi(x)^{i_1}\dot{b}_1^{j_1}\varphi(x)^{i_2}\dot{b}_2^{j_2}\cdots\cdots\cdots\varphi(x)^{i_n}\dot{b}_n^{j_n},$$

where each $\dot{b}_k$ is in B and every i_k and j_k are non-negative integers for $k=1, 2, \cdots, n$. Taking account of each $\varphi(x)^j$ and b^i are cosets of G/H, we see that a is an element of $\{x, \varphi^{-1}(B)\}$. Accordingly, $\varphi^{-1}(B)$ is a normal divisor of $\varphi^{-1}(A)$, which implies that B is also normal in A. This completes the proof.

THEORM. *Let G be a locally compact topological $\dot{N}$-group and H a closed maximal subgroup of G, then the quotient group G/H is an an abstract $\dot{N}$-group.*

PROOF. If H coincide with G, then there is nothing to prove. Suppose that H is a proper subgroup of G, there exist an element x in G such that $\{x, H\}$ is G. Since G is a countable union of closed sets, namely $G=\bigcup x^iH$, H must be open in G. It follows that G/H is a discrete topological $\dot{N}$-group by the preceeding Lemma, and this proves the theorem.

COROLLARY. *If G is a compact $\dot{N}$-group then any quotient group of G with respect to a closed maximal subgroup is a finite group.*

References

[1] V. P. Platonov, *Locally projective nilpotent topological groups and groups with normalizer condition*, Dokl. Akad. Nauk BSSR 8(1964).

[2] V. I. Ušakov, *Groups with normalizer condition*, English transl ., Amer. Math. Soc. Transl. (2) 82(1969).

[3] ―――――, *Classes of conjugate subgroups in topological groups*, English transl., Dokl. Akad. Nauk SSSR Tom 190, No. 1(1970).

Seoul National University