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A note on the well-formed formulas of a pure functional calculus of the first order

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Abstract

The purpose of this note is to use primitive recursive functions and predicates similar to those presented by Davis [2] to show the well-known result first proved by Godel in [3] that the well-formed formulas of a pure functional calculus form a recursive set. The technique shown in this paper immediately suggests a unique approach to Theorem proving using a computer.

1. Definitions

A pure functional calculus of the first order, F^{1p}, is defined by Church [1] to be the logistic system having as its primitive symbols the eight improper symbols

$$[\supset]\sim(,)\ V$$

and the infinite list of individual variables

$$x\ y\ z\ x_1\ y_1\ z_1\ x_2\ y_2\ z_2 \ldots\ldots$$

Also some or all of the following, including either at least one of the infinite lists of functional variables or at least one functional constant, are included: the infinite list of propositional variables

$$p\ q\ r\ s\ p_1\ q_1\ r_1\ s_1\ p_2 \ldots\ldots$$

and, for each positive integer n, an infinite list of n-ary functional variables, namely, the infinite list of singularly functional variables

$$F^1\ G^1\ H^1\ F_1^1\ G_1^1\ H_1^1\ F_2^1 \ldots\ldots$$

the infinite list of binary functional variables

$F^2\ G^2\ H^2\ F_1^2\ G_1^2\ H_1^2\ F_2^2$...........and so on

including any number of individual constants, any number of singularly functional constants, binary functional constants, ternary functional constants, etc.

The formation rules of F^{1p} are given by Church [1] as:

(1) A propositional variable standing alone is a welll-formed formula (henceforth referred to as *wff*).

(2) If f is an n-ary functional variable or an n-ary functional constant, and if $a_1,\ a_2,\ \ldots,\ a_n$ are individual variables or individual constants or both (not necessarily all different), then $f(a_1, a_2, \ldots, a_n)$ is a *wff*.

(3) If Γ is a *wff* then $\sim\Gamma$ is a *wff*.

(4) If Γ and Δ are *wff* then $[\Gamma \supset \Delta]$ is a *wff*.

(5) If Γ is a *wff* and a is an individual variable, then $(\forall a)\ \Gamma$ is a *wff*.

We also define a number of functions and predicates suggested by Davis (1958):

D1. The function $x \dot{-} y$ is defined by

$$x \dot{-} y = \begin{Bmatrix} x-y & \text{if} & x \geqq y \\ 0 & \text{if} & x < y \end{Bmatrix}$$

D2. If $P(y,\ x^{(n)})$ is an $(n+1)$-ary predicate then

$$f(z, x^{(n)}) = \overset{z}{\underset{y=0}{\Phi}} P(y, x^{(n)})$$

is understood to be the $(n+1)$-ary total function that satisfies the equation

$$f(z, x^{(n)}) = \min_y [y \leqq z \wedge P(y, x^{(n)})]$$

where this is defined and

$$f(z, x^{(n)}) = 0$$

elsewhere.

D3. The function $Pr(n)$ is defined as the nth prime in order of magnitude, where we arbitrarily take the 0th prime equal to 0, namely

$$Pr(0) = 0$$

$$Pr(n+1) = \overset{Pr(n)!+1}{\underset{y=0}{\Phi}} [Prime(y) \wedge y > Pr(n)]$$

where $Prime(x)$ states that x is a prime number.

D4. Let $P(x_1, \cdots, x_n)$ be an n-ary predicate. Then, by the extension of P, written

$$\{x_1, x_2, \ldots, x_n | P(x_1, x_2, \ldots, x_n)\}.$$

we shall mean set of all n-tuples $(a_1, a_2, \ldots, a_n)$ for which $P(a_1, a_2, \ldots, a_n)$ is true.

D5. The predicate Gl is defined by the equation

$$n\ Gl\ x = \overset{x}{\underset{y=0}{\Phi}} [(Pr(n)^y | x) \wedge \sim (Pr(n)^{y+1}) | x)]$$

(If x is the Godel number of M, where M consists of the symbols $\gamma_1, \gamma_2, \cdots \gamma_p$, then if $0 < n \leqslant p$, $n\ Gl\ x$ is the number associated with γ_n, whereas if $n=0$ or $n > p$, then $n\ Gl\ x = 0$.)

D6. The function $L(x)$ is defined by the equation

$$L(x) = \overset{x}{\underset{y=0}{\Phi}} [(y\ Glx > 0) \frown \overset{x}{\underset{i=0}{\bigwedge}} (y+i+1)\ Gl\ x = 0]$$

D7. The expression (x/y) states that y divides x evenly.

2. The recursiveness of the *wff* of $F^1{}_p$

Using the above functions and predicates it is possible to demonstrate that the *wff* of $F^1{}_p$ form a recursive set. We proceed to arithmetize the formal system as follows:

For the primitive symbols of $F^1{}_p$, we make the correspondences:

[	⊃	]	∼	(	,	)	⊬
↕	↕	↕	↕	↕	↕	↕	↕
2	3	4	5	6	7	8	9

For the infinite list of individual variables (which can be denumerably infinite in number), we make the correspondences:

$$\begin{array}{ccccccccc} x & y & z & x_1 & y_1 & z_1 & x_2 & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots\cdots \\ 11 & 11^2 & 11^3 & 11^4 & 11^5 & 11^6 & 11^7 & \end{array}$$

For the individual constants:

$$\begin{array}{cccccccc} f & g & h & f_1 & g_1 & h_1 & f_2 & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots\cdots \\ 13 & 13^2 & 13^3 & 13^4 & 13^5 & 13^6 & 13^7 & \end{array}$$

For the propositional variables:

$$\begin{array}{ccccccccc} p & q & r & s & p_1 & q_1 & r_1 & s_1 & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots\cdots \\ 17 & 17^2 & 17^3 & 17^4 & 17^5 & 17^6 & 17^7 & 17^8 & \end{array}$$

For the singularly functional variables:

$$\begin{array}{cccccccc} F^1 & G^1 & H^1 & F_1{}^1 & G_1{}^1 & H_1{}^1 & F_2{}^1 & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots\cdots \\ 19 & 19^2 & 19^3 & 19^4 & 19^5 & 19^6 & 19^7 & \end{array}$$

For the singularly functional constants:

$$\begin{array}{ccccccc} P^1 & Q^1 & R^1 & S^1 & P_1{}^1 & Q_1{}^1 & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots\cdots \\ 23 & 23^2 & 23^3 & 23^4 & 23^5 & 23^6 & \end{array}$$

For the binary functional variables:

$$\begin{array}{cccccc} F^2 & G^2 & H^2 & F_1{}^2 & G_1{}^2 & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots\cdots \\ 29 & 29^2 & 29^3 & 29^4 & 29^5 & \end{array}$$

For the binary functional constants:

$$\begin{array}{ccccccc} P^2 & Q^2 & R^2 & S^2 & P_1{}^2 & Q_1{}^2 & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots\cdots \\ 31 & 31^2 & 31^3 & 31^4 & 31^5 & 31^6 & \end{array}$$

For the n-ary functional variables:

$$\begin{array}{ccccc} F^n & G^n & H^n & F_1{}^n & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots\cdots \\ Pr(7+n) & [Pr(8+n)]^2 & [Pr(8+n)]^3 & [Pr(8+n)]^4 & \end{array}$$

Finally, for the n-ary functional constants, we make the correspondences:

$$\begin{array}{cccc} P^n & Q^n & R^n & S^n \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \cdots\cdots \\ Pr(8+n) & [Pr(8+n)]^2 & [Pr(8+n)]^3 & [Pr(8+n)]^4 \end{array}$$

Thus, any formula in F^{1p} will correspond a sequence of numbers, namely those corresponding to the successive symbols of the formula. For example, to $[(\forall x)F^1(x)\supset\sim(\forall y)G^1(y)]$ will correspond the sequence

2 6 9 11 8 19 6 11 8 3 5 6 9 11^2 8 19^2 6 11^2 8 4

We can make a unique number correspond to the formula by taking the product of the successive prime numbers (in their natural order) with powers equal to the numbers of the symbols (in the order in which they occur).

Thus to the formula above will correspond the number

$$2^2\ 3^6\ 5^9\ 7^{11}\ 11^8\ 13^{19}\ 17^6\ 19^{11}\ 23^8\ 29^3$$
$$31^5\ 37^6\ 43^9\ 47^{11^2}\ 53^8\ 59^{19^2}\ 61^6\ 67^{11^2}\ 71^8\ 73^4$$

We call this the Godel number of the formula, or, if M is an expression consisting of the symbols $\Gamma_1,\ \Gamma_2,\ \ldots,\ \Gamma_{n-1},\ \Gamma_n$ and we let $a_1,\ \ldots,\ a_n$ be the integers associated with these symbols, then the Godel number of M is the integer

$$r=\prod_{k=1}^{n} Pr(k)^{a_k} \qquad \text{and we write} \qquad gn(M)=r$$

If M is empty we write $gn(M)=1$

We will define $Welf(x)$ to mean x is the Godel number of a well-formed -formula.

We will now define five intermediate predicates which will correspond to the formation rules of F^{1p}, (1), (2), (3), (4), and (5), respectively.

(1′) $Prop(x)$ holds if and only if

$$\bigwedge_{m=1}^{\min_y[1\ Gl\ x\dot{-}17_y<\max\ 17]} [(L(x)=1)\wedge(17m/l\ Gl\ x)]$$

(x is the Godel number of a propositional variable standing alone.)

(2′) $Func(x)$ holds if and only if

$$\left[\left\{\bigvee_{n=1}^{5L(x)}\ \bigwedge_{m_1=1}^{A_1}\ \bigwedge_{m_2=1}^{A_2}\ \bigwedge_{m_3=1}^{A_3}\ \bigwedge_{m_4=1}^{A_4}\left[\bigwedge_{i=1}^{6} V_i\right]\right\}\smile\right.$$
$$\left.\left\{\bigwedge_{m=1}^{A_5}\ \bigwedge_{m=1}^{A_6}\ \bigwedge_{m=1}^{A_7}\ \bigwedge_{m=1}^{A_8}\left[\bigwedge_{i=7}^{11} V_i\right]\right\}\right]$$

where

$$A_1=\min_y\ [1\ Gl\ x\dot{-}yPr(9+n)<Pr(9+n)]$$
$$A_2=\min_y\ [1\ Gl\ x\dot{-}yPr(10+n)<Pr(10+n)]$$
$$A_3=\min_y\ [(2i+1)\ Gl\ x\dot{-}11y<11]$$
$$A_4=\min_y\ [(2i+1)\ Gl\ x\dot{-}13y<13]$$
$$A_5=\min_y\ [1\ Gl\ x\dot{-}19y<19]$$
$$A_6=\min_y\ [1\ Gl\ x\dot{-}23y<23]$$
$$A_7=\min_y\ [3\ Gl\ x\dot{-}11y<11]$$
$$A_8=\min_y\ [3\ Gl\ x\dot{-}13y<13]$$
$$V_1=(L(x)=2(n+1))$$
$$V_2=([m_1Pr(9+n)/1\ Gl\ x]\smile[m_2Pr(10+n)/1\ Gl\ x])$$
$$V_3=(2\ Gl\ x=6)$$
$$V_4=([2n+2]\ Gl\ x=8)$$
$$V_5=\left(\bigwedge_{i=2}^{n}[2i\ Gl\ x=7]\right)$$
$$V_6=\left(\bigwedge_{z=1}^{n}[(11m_3/[2i+1]\ Gl\ x)\smile(13m_4/[2i+1]\ Gl\ x]\right)$$
$$V_7=(L(x)=4)$$
$$V_8=([19m_5/1\ Gl\ x]\vee[23m_6/1\ Gl\ x])$$
$$V_9=(2\ Gl\ x=6)$$
$$V_{10}=([11m_7/3\ Gl\ x]\vee[13m_8/3\ Gl\ x])$$
$$V_{11}=(4\ Gl\ x=8)$$

This states that x is a Godel Number of any n-ary functional variable or constant standing alone (with its argument).

(3′) *C.V. Neg*(x) holds if and only if

$$[[Welf(y)]\wedge[x=2^5*y])$$

(If Γ is *wf* then $\Gamma\sim$is *wf*)

(4′) *C.V. Impl*(x) holds if and only if

$$[[Welf\ (y)]\wedge[Welf\ (z)]\wedge[x=2^2*y*[Pr[L(y)+2]]^3*z*\ [Pr[L(y)+L(z)+3]]^4]]$$

(5′) *C.V. Quant*(x) holds if and only if

$$[Welf(y)\wedge\left\{\min_y \bigwedge_{m=1}^{[11y=n]} [[x=2^6*3^9*5^n*7^8*y]\wedge[11m/n]]\right\}$$

We now make the definition:

Welf(x) holds if and only if

$$[[Prop(x)]\vee[Func(x)]\vee[C.V.Neg(x)]\vee[C.V.Impl(x)]\vee[C.V.Quant(x)]]$$

We note that *Welf*(x) is recursive. But this simply states that the *wff* of F^1p form a recursive set.

References

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3. Kurt Gödel, Über formal Unentscherdbar Sätze der Principia Mathematica und verwandter Systeme I, *Monatshefte für Mathematik und Physik*. Vol. 38 (1931) pp. 178—198.

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