C^*-continuous functions

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1. Introduction

In this paper, we introduce the concept of c^*-continuous functions and investigate some properties of these functions and obtain some interesting results.

DEFINITION 1. Let X and Y be topological spaces. The function $f: X\to Y$ is c^*-continuous iff C is a countably compact and closed subset of Y, then $f^{-1}(C)$ is a closed subset of X. Equivalently, $f: X\to Y$ is c^*-continuous iff U is an open subset of Y such that $Y\sim U$ is a countably compact, then $f^{-1}(U)$ is an open subset of X.

Another equivalence is same as following: $f: X\to Y$ is c^*-continuous iff for each $x\in X$, if U is an open subset of containing $f(x)$ such that $Y\sim U$ is countably compact, then there exists an open subset V of x in X with $f(V)\subset U$. We clearly note that if $f: X\to Y$ is c^*-continuous and $A\subset X$, then $f|A: A\to Y$ is c^*-continuous.

DEFINITION 2[2]. Let X and Y be topological spaces, let $f: X\to Y$ be a function, and let $p\in X$. Then f is said to be c^*-continuous at p provided if U is an open subset of Y containing $f(p)$ such that $Y\sim U$ is compact, then there is an open subset V of X containing p such that $f(V)\subset U$. The function f is said to be c-continuous (on X) provided f is c-continuous at each point of X. Here, we note that every continuous function is c^*-continuous function and every c^*-continuous function is c-continuous function. But the inverse of these implications is not always true.

EXAMPLE 1a. Let $[0,\Omega]$ be the ordinal space, let $[0,\Omega)\cup\{\Omega\}$ be the free union of disjoint subspace $[0,\Omega)$ and $\{\Omega\}$ of the ordinal space $[0,\Omega]$.

Define a function $f: [0,\Omega]\to[0,\Omega)\cup\{\Omega\}$ by $f(\alpha)=\alpha$ for each $\alpha\in[0,\Omega]$. Then f is c-continous but not c^*-continuous. For, since f is continuous at every point except Ω, it is sufficient to show that f is c-continuous at Ω. Let V be an open subset of $[0,\Omega)\cup\{\Omega\}$ containing $f(\Omega)=\Omega$ such that $[0,\Omega)\cup\{\Omega\}\sim V$ is compact. Thus V^c does not contain Ω and $V^c\subset\bigcup_\alpha\{[0,\alpha)\mid\alpha<\Omega\}$ for each $\alpha<\Omega$. Since V^c is compact, $V^c\subset[0,\alpha_0)\subset[0,\alpha_0]$ for some $0<\alpha_0<\Omega$. Thus $V\supset(\alpha_0,\Omega]$. Then $(\alpha_0,\Omega]$ is an open set containing Ω and $f((\alpha_0,\Omega])=(\alpha_0,\Omega]\subset V$. Hence f is c-continuous at Ω. But, clearly f is not c^*continuous.

EXAMPLE 1b. Let R be the reals with the usual topology and define $f: R\to R$ by

$$f(x)=\begin{cases}\dfrac{1}{x} & \text{if } x\neq 0,\\[2mm] \dfrac{1}{2} & \text{if } x=0.\end{cases}$$

Then f is c^*-continuous but not continuous (cf. [1]). Throughout this paper, locally countably compactness assumes to be a T_2-space and each point has a relatively countably compact neighborhood.

Definitions and notations follow Dugundji [4].

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2. Main results

THEOREM 1. *If X and Y are topological spaces and either* (1) $X=\bigcup_{\alpha\in\Gamma} A_\alpha$, *where each A_α is an open subset of X or* (2) $X=\bigcup_{i=1}^{n} B_i$, *where each B_i is a closed* subset *of X and $f: X\to Y$ is a function such that either each $f|A_\alpha$ or each $f|B_i$ is c^*-continuous, then f is c^*-continuous.*

Proof. Ad(1). Assume that $X=\bigcup_{\alpha\in\Gamma}$, where each A_α is an open subset of X. Let U be an open subset of Y such that $Y\sim U$ is a countably compact. Then $f^{-1}(U)=\bigcup_{\alpha}(f|A_\alpha)^{-1}(U)$. Since each $f|A_\alpha$ is c^*-continuous, $f^{-1}(U)$ is an open subset of X.

Ad(2). Let C be a countably compact and closed subset of Y. Then $f^{-1}(C)=\bigcup_{i=1}^{n}(f|B_i)^{-1}(C)$. Since each $f|B_i$ is c^*-continuous, $f^{-1}(C)$ is a closed subset of X.

DEFINITION 3. A mapping $f: X\to Y$ from one topological space onto another is compact if for every compact set $K\subset Y$, $f^{-1}(K)$ is compact. [2]

THEOREM 2. *Let $f: X\to Y$ be a compact mapping, where X is T_2-space and Y is paracompact. Then f is c^*-continuous.*

Proof. Let C be a countably compact and closed subset of Y. Since Y is paracompact space, C is compact subset of Y. Since f is a compact mapping $f^{-1}(C)$ is a compact subset of *a* T_2-space X. Hence $f^{-1}(C)$ is a closed subset of X.

LEMMA 3. *Let $f: X\to Y$ be a function such that $G(f)$ is closed, and X be the first-countable space. If K is a countably compact subset of Y, then $f^{-1}(K)$ is a closed subset of X.*

Proof. Let K be a countably compact subset of Y. Suppose $f^{-1}(K)$ is not closed. Since X is the first-countable space, there is a point p in $X\sim f^{-1}(K)$ and a sequence $\{x_n\}$ in $f^{-1}(K)$ such that $x_n\to p$. Since K is a countably compact, a sequence $\{f(x_n)\}$ in K accumulates to some q in K. Thus $(x_n, f(x_n))$ accumulates to (p, q) so that $p\in f^{-1}(q)\subset f^{-1}(K)$. But it contradicts to p in $X\sim f^{-1}(K)$.

THEOREM 4. *Let $f: X\to Y$ be a function having a closed graph $G(f)$ and X be the first-countable space. Then f is c^*-continuous.*

Proof. It is clear by lemma 3.

THEOREM 5. *Let $f: X\to Y$ be a c^*-continuous and Y be a countably compact space. Then f is continuous.*

Proof. It is clear from the definition of c^*-continuous.

REMARK. Let $f: X\to Y$ be a function and Y be a compact space. Then c-continuous, c^*-continuous and continuous are equivalent.

THEOREM 6. *Let $f: X\to Y$ be c^*-continuous, where X is a saturated space and Y is a locally countably compact regular space. Then f is continuous.*

Proof. Let $x\in X$ and let W be an open subset of Y conaining $f(x)$. Since Y is regular, there is an open set U such that $f(x)\in U\subset\bar{U}\subset W$. Let $y\in Y\sim\bar{U}$. Since Y is regular, there is an open set V_y such that $y\in V_y\subset\bar{V}_y\subset Y\sim\bar{U}$. Thus $V_y\cap\bar{U}=\phi$. Since Y is a locally countably compact regular space, there exists a countably compact set C_y such that $y\in C_y\subset \bar{C}_y\subset V_y$. Thus $\bar{C}_y\cap U=\phi$. Then $Y\sim C_y$ is an open set containing $f(x)$ such that $Y\sim\bar{C}_y$ has countably compact complement. Since f is c^*-continuous there exists an open set N_y of x such that $f(N_y)\subset Y\sim\bar{C}_y$. Let $N=\cap\{N_y|y\in Y\sim\bar{U}\}$. Since X is a saturated space, N is open and N contains x. Thus $f(N)\subset\bar{U}\subset M$. Hence f is continuous.

LEMMA 7. *Let $f: X\rightarrow Y$ be a function, where X and Y are topological spaces. Then $G(f)$ is closed if and only if for each $x\in X$ and $y\in Y$, where $y\neq f(x)$, there exist open sets U and V containing x and y respectively such that $f(U)\cap V=\phi$.* ([3], lemma 1.)

LEMMA 8. *Let $f: X\rightarrow Y$ be a c^*-continuous and Y be a locally countably compact regular space. Then $G(f)$ is closed.*

Proof. For each $x\in X$, and $y\in Y$, where $y\neq f(x)$ since Y is a T_2-space, there is an open set U of y such that $U\not\ni f(x)$. Since Y is a locally countably compact regular space, there is an open set V such that $y\in V\subset\bar{V}\subset U$. Then $Y\sim\bar{V}$ is an open set of Y with countably compact complement. Since f is c^*-coutinuous, $f^{-1}(Y\sim\bar{V})=X\sim f^{-1}(\bar{V})$ is an open set of X. By lemma 7, $G(f)$ is closed.

THEOREM 9. *Let X be the first-countable space and Y be a locally countably compact regular space. Then $f: X\rightarrow Y$ is c^*-continuous if and only if $G(f)$ is closed.*

Proof. It is clear from theorem 4 and lemma 8.

THEOREM 10. *Let X be a Baire space and Y be the first countable space such that $Y=\bigcup_{j=1}^{\infty}C_j$, where each C_j is a countable compact subset of Y, let $f: X\rightarrow Y$ be a c^*-continuous. Then f is continuous on a dense subset of X.*

Proof. Let U be a nonempty open subset of a Baire space X. Then U is a Baire space. Suppose that f is not continous at any point of U. Let $F_n=\{x|x\in U$ and $f(x)\in Y\sim\bigcup_{j=1}^{n}C_j\}$ for each positive integer n. Since Y is the first-countable, $\bigcup_{j=1}^{n}C_j$ is a closed and countably compact subset of Y. Then $Y\sim\bigcup_{j=1}^{n}C_j$ is an open set in Y with a countably compact complement. Since f is c^*-continuous, $f^{-1}(Y\sim\bigcup_{j=1}^{n}C_j)$ is an open set in X. Hence $F_n=f^{-1}(Y\sim\bigcup_{j=1}^{n}C_j)\cap U$ is an open subset of U. Suppose that F_n is not a dense subset of U. Then there exists some nonempty open subset V of U containing no point of F_n. Clearly, $f(V)\subset\bigcup_{j=1}^{n}C_j$. Since $f|V$ is continuous, f is continuous at a point of U. It is a contradiction. Hence F_n is dense in U. Since each F_n is open and dense in a Baire space U, $\bigcap_{j=1}^{\infty}F_j$ is dense in U. But $\bigcap_{j=1}^{\infty}F_j=\phi$. It is continuous on a dense subset of X.

THEOREM 11. *Let $f: X\rightarrow Y$ be bijection and continuous function and Y be the first-countable space. Thus $f^{-1}: Y\rightarrow X$ is c^*-continuous.*

Proof. Let C be a countably compact and closed subset of X. Since f is continuous, $f(C)$ is a countably compact subest of the first-countable space Y. Thus $f(C)$ is a closed subset of X. Hence f^{-1} is c^*-continuous.

The following statement is in ([4], p. 253) and directly proved. But using the concept of c^*-continuous, we can prove this statement.

COROLLARLY 12. *Let $f: X \to Y$ be bijection and continuous function and Y be the first-countable space and X be countably compact. Then f is a homeomorphism.*

Proof. By Theorem 11, f^{-1} is c^*-continuous. Since X is countably combact by Theorem 5, f^{-1} is continuous. Hence f is a homeomorphism.

References

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