On the Fourier series of a harmonizable process

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1. Introduction

The harmonizable process was first defined by M. Loève [11], [12], as a nonstationary generalization of the stationary process. T. kawata [7], [8], studied on the Fourier series of the stationary process. This paper mainly deals with analogues of some results of the above study to the case of the harmonizable process.

Throughout this paper, without otherwise mentioned, a harmonizable process is denoted by $X(t)$, $-\infty<t<\infty$; while a weakly stationary process by $x(t)$, $-\infty<t<\infty$. The following notations and properties are always assumed.

(i) In the case of the harmonizable process:

$$EX(t)=0, \qquad \text{for all } t\in(-\infty,\infty),$$

$$E|X(t)|^2<\infty, \qquad \text{for all } t\in(-\infty,\infty),$$

(1.1) $$X(t)=\int_{-\infty}^{\infty} e^{it\lambda}dY(\lambda) \qquad \text{a.s.},$$

where

$$EY(\lambda)=0 \text{ and } E|Y(\lambda)|^2<\infty \text{ for all } \lambda\in(-\infty,\infty)$$

such that the covariance of $Y(\lambda)$ and $Y(\lambda')$ represented by

$$EY(\lambda)\bar{Y}(\lambda')=F(\lambda,\lambda')$$

is of bounded variation on the two dimensional (λ,λ')-plane and has the property:

(1.2) $$R(t,t')=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(t\lambda-t'\lambda')}d^2F(\lambda,\lambda'),$$

where $R(t,t')=EX(t)\bar{X}(t')$.

We call $F(\lambda,\lambda')$ the spectral function corresponding to the $X(t)$-process, and (1, 1) and (1, 2) are respectively called spectral representation of harmonizable process $X(t)$ and spectral representation of harmonizable covariance $R(t,t')$.

(ii) In the case of the weakly stationary process:

$$Ex(t)=0, \qquad \text{for all } t\in(-\infty,\infty),$$

$$E|x(t)|^2<\infty, \qquad \text{for all } t\in(-\infty,\infty),$$

(1.3) $$X(t)=\int_{-\infty}^{\infty} e^{it\lambda}dy(\lambda) \qquad \text{a.s.},$$

where $y(\lambda)$ is a process with orthogonal increments such that

$$Ey(\lambda)=0,\ E|y(\lambda)|^2=F(\lambda)<\infty,\ E|dy(\lambda)|^2=dF(\lambda)$$

and

$$r(t)=Ex(t+u)\bar{x}(u)=\int_{-\infty}^{\infty} e^{it\lambda}dF(\lambda).$$

Here the spectral function $F(\lambda)$ is real, nondecreasing and bounded.

For the convenience of the subsequent study we shall mention the following fact which is already known:

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LEMMA 1-A.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2F(\lambda,\lambda')\geqq 0. \tag{1.4}$$

In fact, from (1.2) we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2F(\lambda,\lambda')=E|X(0)|^2\geqq 0.$$

We remark here that without changing the value of the integral (1.1), we can always suppose that $Y(\lambda)$ is everywhere continuous to the right in quadratic mean, so that $Y(\lambda+0)=Y(\lambda)$. The function $F(\lambda,\lambda')$ then defines a complex valued mass distribution over the whole two dimensional Euclidean (λ,λ')-plane such that the mass carried by any rectangle $h<\lambda<h+\Delta h$, $h'<\lambda'<h'+\Delta h'$ is equal to the second order difference $\Delta^2F(\lambda,\lambda')$ corresponding to this rectangle. It follows from the Hermitean symmetry of covariance that the masses carried by two sets of points symmetrically situated with respect to the diagonal $\lambda=\lambda'$ of (λ,λ')-plane are always complex conjugates.

The Fourier coefficients of $X(t)$ and $x(t)$ over $(-T/2, T/2)$ are respectively

$$C_n=\frac{1}{T}\int_{-T/2}^{T/2} X(t)e^{-in\omega_0 t}dt, \qquad n=0,\ \pm1,\pm2,\cdots\cdots \tag{1.5}$$

and

$$c_n=\frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-in\omega_0 t}dt, \qquad n=0,\ \pm1,\pm2,\cdots\cdots$$

where T is any positive number and $\omega_0=\frac{2\pi}{T}$.

For the later use we define

$$G(\mu)=\int_{\lambda=-\infty}^{\mu}\int_{\lambda'=-\infty}^{\infty} |d^2F(\lambda,\lambda')|. \tag{1.6}$$

Then $G(\lambda)$ is never decreasing, bounded and non negative function of λ. (See H. Cramér [3] page 73.) It is obvious that the function

$$B(\mu)=\int_{\lambda=-\infty}^{\mu}\int_{\lambda'=-\infty}^{\infty} d^2F(\lambda,\lambda') \tag{1.7}$$

is of bounded variation over $-\infty<\mu<\infty$.

From the Hermitean symmetry of $F(\lambda,\lambda')$, it follows that

$$G(\mu)=\int_{\lambda=-\infty}^{\mu}\int_{\lambda'=-\infty}^{\infty} |d^2F|=\int_{\lambda=-\infty}^{\infty}\int_{\lambda'=-\infty}^{\mu} |d^2F|.$$

$$B(\mu)=\int_{\lambda=-\infty}^{\mu}\int_{\lambda'=-\infty}^{\infty} d^2F$$

$$\bar{B}(\mu)=\int_{\lambda=-\infty}^{\infty}\int_{\lambda'=-\infty}^{\mu} d^2F.$$

2. The behavior of Fourier coefficients as $n\to\infty$.

T. Kawata [8] has shown the following theorem for the weakly stationary process.

THEOREM 2—A. (i) *If for* $0\leq\alpha<1$

$$\int_{-\infty}^{\infty} |\lambda|^{\alpha}dF(\lambda)<\infty, \tag{2.1'}$$

then

(2.2)′ $$\sum_{|n|>N} E|c_n|^2 = o(N^{-\alpha})$$

as $N \to \infty$.

(ii) *If* $x(t)$ *is periodic with period* T, *then under the condition* (2.1)′ *with* $0 \leq a < 2$, (2.2)′ *holds. And if* $\alpha = 2$, *then*

$$\sum_{|n|>N} E|c_n|^2 = O(N^{-2}).$$

(iii) *If* $x(t)$ *is periodic with period* T, *and*

$$\int_{-\infty}^{\infty} |\lambda|(log^{+}|\lambda|)dF(\lambda) < \infty, \qquad \beta > 0,$$

then

$$\sum_{|n|>N} E|c_n|^2 = o(1/N(\log^{\beta} N)).$$

The analogue of the above theorem to the harmonizable process is as follows.

THEOREM 2—1. (i) *If*

(2.1) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\lambda|^{\alpha}|d^2F(\lambda, \lambda')| < \infty,$$

for $0 \leq \alpha < 1$ *then*

(2.2) $$\sum_{|n|>N} E\,|C_n|^2 = o(N^{-\alpha})$$

as $N \to \infty$.

(ii) *If* $X(t)$ *is periodic with period* T, *then under the condition* (2.1) *with* $0 \leq \alpha < 2$, (2.2) *holds, and if* $\alpha = 2$, *then*

(2.3) $$\sum_{|n|>N} E|C_n|^2 = O(N^{-2}).$$

(iii) *If* $X(t)$ *is periodic with period* T *and*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\lambda|(log^{+}|\lambda|)^{\beta}|d^2F(\lambda, \lambda')| < \infty, \quad \beta > 0,$$

then

$$\sum_{|n|>N} E|C_n|^2 = o\ (1/N(log^{\beta} N)).$$

REMARK 2—1. *The condition* (2.1) *can be written by* $\int_{-\infty}^{\infty} |\lambda|^{\alpha} dG(\lambda) < \infty$, *where* $G(\lambda)$ *is defined in* (1.6). *Similarly the condition in* (iii) *can be expressed with respect to* $G(\lambda)$.

(*Proof*) (i)

$$\sum_{|n|>N} E|C_n|^2 = \sum_{n>N} + \sum_{n<-N}.$$

$$\sum_{n>N} = \sum_{n>N}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} dt dt' \frac{1}{T^2} e^{-i(nt-nt')\frac{2\pi}{T}} \cdot$$

$$\cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(t\lambda - t'\lambda')} d^2F(\lambda, \lambda') = \sum_{n>N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2F(\lambda, \lambda') \frac{1}{T}\int_{-T/2}^{T/2} e^{it(\lambda - 2\pi n/T)} dt \cdot$$

$$\cdot \frac{1}{T}\int_{-T/2}^{T/2} e^{-it'(\lambda' - 2\pi n/T)} dt' = \sum_{n>N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sin \pi(n - T\lambda/2\pi)}{\pi(n - T\lambda/2\pi)} \cdot \frac{\sin \pi(n - T\lambda'/2\pi)}{\pi(n - T\lambda'/2\pi)} d^2F(\lambda, \lambda').$$

If we put

$$a_n(\lambda) = \frac{\sin \pi(n - T\lambda/2\pi)}{\pi(n - T\lambda/2\pi)}.$$

then we have

$$\left|\sum_{n>N}\right| \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{n>N} |a_n(\lambda)a_n(\lambda')|\,|d^2F(\lambda,\lambda')|$$

$$\leq \int_{\lambda\leq\frac{\pi N}{T}}\int_{\lambda'\leq\frac{NT}{\pi}} + \int_{\lambda>\frac{\pi N}{T}}\int_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\int_{\lambda'>\frac{\pi T}{N}} = I_1+I_2+I_3.$$

$$I_1 \leq \int_{\lambda\leq\frac{\pi N}{T}}\int_{\lambda'\leq\frac{\pi N}{T}} \Sigma \frac{1}{\pi^2} \frac{1}{n-\frac{T\lambda}{2\pi}}\cdot\frac{1}{n-\frac{T\lambda'}{2\pi}} \left|d^2F(\lambda,\lambda')\right|$$

$$\leq \sum_{m=\left[\frac{N}{2}\right]} \frac{1}{\pi^2}\cdot\frac{1}{m^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|d^2F(\lambda,\lambda')\right| = O(N^{-1}).$$

$$I_2 = \int_{\lambda>\frac{\pi N}{T}}\int_{-\infty}^{\infty}\sum_{n>N}|a_n(\lambda)a_n(\lambda')|\ \ |d^2F(\lambda,\lambda')|.$$

We must prove here that

1) $\sum_{n>N}|a_n(\lambda)a_n(\lambda')| = O(1)$ uniformly for λ, λ',

2) $\int_{\lambda>\frac{\pi N}{T}}\int_{-\infty}^{\infty}|d^2F(\lambda,\lambda')| = o(N^{-\alpha})$.

The proof of 1): The number of n's which come into the interval $\left(\frac{T\lambda}{2\pi}-1,\ \frac{T\lambda}{2\pi}+1\right)$ or $\left(\frac{T\lambda'}{2\pi}-1,\ \frac{T\lambda'}{2\pi}+1\right)$ is 4, and the summation of $|a_n(\lambda)a_n(\lambda')|$ with respect to this four n's is less than 4, because $|a_n(\lambda)a_n(\lambda')|\leq 1$ for all n Therefore we have only to show that the summation of $|a_n(\lambda)a_n(\lambda')|$ with respect to n's which are not in the intervals is dominated by a constant independent of λ and λ'. Now we have

$$\left|a_n(\lambda)a_n(\lambda')\right| \leq \frac{1}{\pi^2}\left|\frac{1}{\frac{T\lambda}{2\pi}-n}\cdot\frac{1}{\frac{T\lambda'}{2\pi}-n}\right|.$$

Putting

$$\min\left(\left[\left|\frac{T\lambda}{2\pi}-n\right|\right],\ \left[\left|\frac{T\lambda'}{2\pi}-n\right|\right]\right) = k_n(\lambda,\lambda'),$$

then for any temporarilly fixed (λ,λ'), the number of n's for which the integer $k_n(\lambda,\lambda')$ assumes the same value is at most four.

Therefore

$$\sum_{n>N}\left|a_n(\lambda)a_n(\lambda')\right| \leq \Sigma\frac{1}{\pi^2}\left|\frac{1}{\frac{T\lambda}{2\pi}-n}\cdot\frac{1}{\frac{T\lambda'}{2\pi}-n}\right| \leq \frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2} \leq \frac{4}{\pi^2},$$

$$n \notin \left(\frac{T\lambda}{2\pi}-1,\ \frac{T\lambda}{2\pi}+1\right)$$

and $$n \notin \left(\frac{T\lambda'}{2\pi}-1,\ \frac{T\lambda'}{2\pi}+1\right),$$

which proves 1).

The proof of 2): We have

$$\int_{\lambda>\frac{\pi N}{T}}\int_{-\infty}^{\infty}|d^2F(\lambda,\lambda')| \leq \left(\frac{T}{\pi N}\right)^{\alpha}\int_{\lambda\geq\frac{\pi N}{T}}\int_{-\infty}^{\infty}|\lambda|^{\alpha}|d^2F(\lambda,\lambda')|$$

$$=\left(\frac{T}{\pi N}\right)^{\alpha}\int_{\lambda>\frac{\pi N}{T}} |\lambda|^{\alpha} dG(\lambda)=o(N^{-\alpha}),$$

which proves 2).

Therefore $I_2=\int_{\lambda>\frac{\pi N}{T}}\int_{-\infty}^{\infty} O(1)\,|d^2F(\lambda,\lambda')|=o(N^{-\alpha})$.

Similarly we can prove that

$$\sum_{n<-N} E|C_n|^2=o(N^{-\alpha}).$$

(ii) From the Parseval relation (see N. Bary [1] page 155),

$$\frac{1}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T}\left|X\left(t+\frac{1}{2}h\right)-X\left(t-\frac{1}{2}h\right)\right|^2 dt=4\sum_{n=-\infty}^{\infty}\left|C_n\right|^2\sin^2\frac{\pi nh}{T}.$$

$$E\left(\frac{1}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T}\left|X\left(t+\frac{1}{2}h\right)-X\left(t-\frac{1}{2}h\right)\right|^2 dt\right)=\frac{1}{T}\int_{-\frac{1}{2}}^{\frac{1}{2}T}\cdot$$

$$\cdot\, dt\left\{R\left(t+\frac{h}{2},\ t+\frac{h}{2}\right)-R\left(t+\frac{h}{2},\ t-\frac{h}{2}\right)-R\left(t-\frac{h}{2},\ t+\frac{h}{2}\right)+R\left(t-\frac{h}{2},\ t-\frac{h}{2}\right)\right\}$$

$$=\frac{1}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} dt\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{it(\lambda-\lambda')}\left(e^{i\frac{h(\lambda-\lambda')}{2}}-e^{i\frac{h(\lambda+\lambda')}{2}}+e^{-i\frac{h(\lambda-\lambda')}{2}}-e^{-i\frac{h(\lambda+\lambda')}{2}}\right)d^2F(\lambda,\lambda')$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin\frac{T}{2}(\lambda-\lambda')}{\frac{T}{2}(\lambda-\lambda')}\cdot 4\cdot\sin\frac{h\lambda}{2}\cdot\sin\frac{h\lambda'}{2}\,d^2F(\lambda,\lambda'),$$

which is dominated by

$$4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\sin\frac{h\lambda}{2}\sin\frac{h\lambda'}{2}\right|\ \left|d^2F(\lambda,\lambda')\right|.$$

By the Schwarz's inequality we have

$$\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\sin\frac{h\lambda}{2}\sin\frac{h\lambda'}{2}\right|\left|d^2F(\lambda,\lambda')\right|\right)^2$$

$$\leq\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin^2\frac{h\lambda}{2}\,|d^2F(\lambda,\lambda')|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin^2\frac{h\lambda'}{2}\,|d^2F(\lambda,\lambda')|$$

$$=\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin^2\frac{h\lambda}{2}\,|d^2F(\lambda,\lambda')|\right)^2=\left(\int_{-\infty}^{\infty}\sin^2\frac{h\lambda}{2}\,dG(\lambda)\right)^2,$$

where we used the Hermitean symmetry of $F(\lambda,\lambda')$.

Therefore

$$4\sum_{n=-\infty}^{\infty}E|C_n|^2\sin^2\frac{\pi nh}{T}\leq 4\int_{-\infty}^{\infty}\sin^2\frac{h\lambda}{2}\,dG(\lambda).$$

Integrating with respect to h over $(0, T/(2N))$, we have

$$\int_0^{T/(2N)} dh\int_{-\infty}^{\infty}\sin^2\frac{h\lambda}{2}\,dG(\lambda)\geq\sum_{|n|>N}E|C_n|^2\int_0^{T/(2N)}\sin^2\frac{\pi nh}{T}\,dh.$$

From $\int_{-\infty}^{\infty}|\lambda|^{\alpha}dG(\lambda)<\infty$, $\quad 0\leq\alpha<2$. $\quad$ it follows that

$$\int_{-\infty}^{\infty}\sin^2\frac{h\lambda}{2}\,dG(\lambda)=o(h^{\alpha})\qquad(h\to 0).$$

(See T. kawata [8], Lemma 1, (i)). From

$$\int_0^{T/(2N)} \sin^2 \frac{\pi n h}{T} dh > \frac{T}{8N}$$

(See N. Bary [1] page 156 (2.2)) and the above relation it follows that

$$o\left(\int_0^{T/(2N)} h^\alpha dh\right) = \sum_{|n|>N} E|C_n|^2 \frac{1}{N},$$

that is $\sum_{|n|>N} E|C_n|^2 = o(N^{-\alpha})$ which proves the first half of (ii).

From

$$\int_{-\infty}^{\infty} |\lambda|^2 dG(\lambda) < \infty,$$

it follows that

$$\int_{-\infty}^{\infty} \sin^2 \frac{h\lambda}{2} dG(\lambda) \leq \int_{-\infty}^{\infty} \frac{h^2}{4} \lambda^2 dG(\lambda) = \frac{h^2}{4} \int_{-\infty}^{\infty} \lambda^2 dG(\lambda) = O(h^2),$$

therefore we have $\sum_{|n|>N} E|C_n|^2 = O(h^2)$, which proves the rest of (ii).

The proof of (iii) follows from that of [8] page 26. (See also [4] Lemma 1 in page 174 and [4] Lemma 7 in page 181).

3. The mean convergence of the Fourier series

THEOREM 3—1. *For every* $|t| < \frac{1}{2}T$,

$$E|S_n(t) - X(t)|^2 \to 0$$

as $n \to \infty$, *where*

$$S_n(t) = \sum_{k=-n}^{n} C_k e^{ik2\pi t/T}.$$

REMARK 3—1. *This theorem which is proved by T. Kawata* [7] *for the stationary process, also holds for the harmonizable process.*

(*Proof*)
$$S_n(t) - X(t) = \frac{1}{T} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} (X(\tau) - X(t)) D_n(T - t) d\tau,$$

where $D_n(T)$ is a Dirichlet kernel, that is,

$$D_n(T) = \frac{\sin\left(n + \frac{1}{2}\right)\frac{2\pi\tau}{T}}{2\sin\frac{\pi\tau}{T}}.$$

We have
$$E|S_n(t) - X(t)|^2 = \frac{1}{T^2} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} d\tau d\tau' \cdot$$

$$\cdot E(X(\tau) - X(t))(\overline{X(\tau') - X(t)}) D_n(\tau - t) D_n(\tau' - t)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^2F(\lambda, \lambda') \cdot \frac{1}{T^2} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} d\tau d\tau' \cdot$$

$$\cdot (e^{i(\lambda\tau - \lambda'\tau')} - e^{i(\lambda t - \lambda'\tau')} - e^{i(\lambda\tau - \lambda' t)} + e^{i(\lambda t - \lambda' t)}) \cdot$$

$$\cdot D_n(\tau - t) D_n(\tau' - t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^2F(\lambda, \lambda') \cdot \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} (e^{i\lambda\tau} - e^{i\lambda t}) D_n(\tau - t) dt \cdot$$

$$\cdot\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}(e^{-i\lambda'\tau'}-e^{-i\lambda' t})D_n(\tau'-t)dt.$$

Since both of the inner Dirichlet integrals are bounded for all λ, λ' and t, and approaches zero as n goes to infinity, our theorem is proved.

4. An approximate Fourier series

T. Kawata [8] constructed an approximate Fourier series of a stationary process using the representation $x(t)=\int_{-\infty}^{\infty}e^{it\lambda}dy(\lambda)$.

We shall here construct an approximate Fourier series of a harmoniozable process using the representation $X(t)=\int_{-\infty}^{\infty}e^{it\lambda}dY(\lambda)$. Let us define

(4.1) $$\hat{C}_n=\int_{\omega_0 n}^{\omega_0(n+1)}dY(\lambda).\quad n=0,\ \pm1,\ \pm2,\cdots\cdots,$$

where $\omega_0=2\pi/T$.

Our approximate Fourier series is defined by

(4.2) $$\hat{X}(t)=\sum_{n=-\infty}^{\infty}\hat{C}_n\,e^{in\omega_0 t}.$$

We shall prove some theorems.

THEOREM 4-1.

$$\sum_{n=-\infty}^{\infty}\hat{C}_n\,e^{in\omega_0 t}$$

converges in L_2-norm.

(*Proof*) We prove that

$$S_{N'}^{N''}=E\left|\sum_{|n|=N'}^{N''}\hat{C}_n\,e^{in\omega_0 t}\right|^2\to 0\qquad (N', N''\to\infty).$$

Since

$$E\,\hat{C}_n\,\overline{\hat{C}}_m=\int_{\omega_0 n}^{\omega_0(n+1)}\int_{\omega_0 m}^{\omega_0(m+1)}d^2F(\lambda,\lambda'),$$

we have

$$\left|S_{N'}^{N''}\right|=\left|\sum_{|n|,|m|=N'}^{N''}e^{i(n-m)\omega_0 t}\int_{\omega_0 n}^{\omega_0(n+1)}\int_{\omega_0 m}^{\omega_0(m+1)}d^2F(\lambda,\lambda')\right|\leq\sum_{|n|,|m|=N'}^{N''}$$
$$\int_{\omega_0 n}^{\omega_0(n+1)}\int_{\omega_0 m}^{\omega_0(m+1)}|d^2F(\lambda,\lambda')|\to 0\qquad (N', N''\to\infty)$$

on account of the fact that $F(\lambda,\lambda')$ is of bounded variation.

THEOREM 4-2.

$$\hat{X}(t)\to X(t)\ \textit{in}\ L_2\textit{-norm as}\ T\to\infty.$$

(*Proof*)

$$E|X(t)-\hat{X}(t)|^2=E|X(t)|^2-EX(t)\overline{\hat{X}(t)}-E\hat{X}(t)\overline{X(t)}+E|\hat{X}(t)|^2=I_1-I_2-I_3+I_4.$$

$$I_1=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{it(\lambda-\lambda')}\,d^2F(\lambda,\lambda'),$$

$$I_4=\sum_{n,n'=-\infty}^{\infty}e^{i(n-n')\omega_0 t}\int_{\lambda=\omega_0 n}^{\omega_0(n+1)}\int_{\lambda'=\omega_0 n'}^{\omega_0(n'+1)}d^2F(\lambda,\lambda'),$$

$$I_2=\sum_{n'=-\infty}^{\infty} e^{in'\omega_0 t}\int_{\lambda=-\infty}^{\infty}\int_{\lambda'=n'\omega_0}^{(n'+1)\omega_0} e^{it\lambda}d^2F(\lambda,\lambda'),$$

$$I_3=\sum_{n=-\infty}^{\infty} e^{in\omega_0 t}\int_{\lambda=\omega_0 n}^{\omega_0(n+1)}\int_{\lambda'=-\infty}^{\infty} e^{-it\lambda'}d^2F(\lambda,\lambda').$$

It is easy to see that I_1, I_2, I_3, I_4 all tend to I_1 as $T\to\infty$. Therefore we have proved that

$$E|X(t)-\hat{X}(t)|^2\to 0$$

as $T\to\infty$

THEOREM 4-3. *Let $g(x)$ be an even function which is non-negative and non-decreasing for $x>0$ and*

(4.4) $$\sum_{n=1}^{\infty}\frac{1}{g(n)}<\infty.$$

If

(4.5) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\lambda)|d^2F(\lambda,\lambda')|<\infty,$$

then (4.2) *absolutely converges almost surely.*

(*Proof*) $$E\sum_{n=-\infty}^{\infty}|\hat{C}_n|=E\sum_{n=-\infty}^{\infty}\left|\int_{\omega_0 n}^{\omega_0(n+1)} dY(\lambda)\right|$$

$$=E\sum_{n=-\infty}^{\infty}\frac{1}{(g(\omega_0 n))^{1/2}}\cdot(g(w_0 n))^{1/2}\left|\int_{\omega_0 n}^{\omega_0(n+1)} dY(\lambda)\right|$$

$$\leq\left(\sum_n\frac{1}{g(\omega_0 n)}\right)^{1/2}\left(\sum_n g(\omega_0 n)\int_{\omega_0 n}^{\omega_0(n+1)}\int_{\omega_0 n}^{\omega_0(n+1)}|d^2F(\lambda,\lambda')|\right)^{1/2}$$

$$\leq\left(\sum_n\frac{1}{g(\omega_0 n)}\right)^{1/2}\left(\sum_n\int_{\omega_0 n}^{\omega_0(n+1)}\int_{\omega_0 n}^{\omega_0(n+1)} g(\lambda)g(\lambda')|d^2F(\lambda,\lambda')|\right)^{1/2}$$

$$\leq\left(\sum_n\frac{1}{g(\omega_0 n)}\right)^{1/2}\left(\int_{-\infty}^{\infty} g(\lambda)dG(\lambda)\right)^{1/2}<\infty. \qquad q.e.d.$$

REMARK 4-1. *The Fourier coefficients of our approximate expansion are not uncorrelated, while the approximate Fourier series of the stationary process in* [8] *has the uncorrelated Fourier coefficients.*

REMARK 4-2. *A. Papoulis* [13] *has shown another approximate Fourier expansion of the stationary process and proved some properties of it, which can be summarized as follows:*

Putting $\omega_0=2\pi/T$,

and defining

$$\check{c}_n=\int_{-\infty}^{\infty} x(t)\frac{\sin(\omega_0 t/2)}{\pi t}e^{-in\omega_0 t}dt$$

and

$$\check{x}(t)=\sum_{n=-\infty}^{\infty}\check{c}_n e^{in\omega_0 t},$$

we have

(1) *$x(t)$ is stationary and m.s. periodic.*

(2) $Ec_n c_m=0$ *when* $n\neq m$,

(3) $\lim_{T\to\infty} E(|x(t)-\check{x}(t)|^2)=0.$

An analogue for the nonstationary processes (including the harmonizable process) is as follows:

Let us define

$$\check{C}_n=\int_{-\infty}^{\infty} X(t)\cdot\frac{\sin(w_0t/2)}{\pi t}e^{-in\omega_0t}dt,$$

and

$$\check{X}(t)=\sum_{n=-\infty}^{\infty}\check{C}_n e^{in\omega_0t}.$$

Through this definition

(2)′ $E\check{C}_n\check{C}_m=0$ *when* $n\neq m$

holds, but

(3)′ $\lim_{T\to\infty} E(|X(t)-\check{X}(t)|^2=0$

never holds.

In other words, in our case, $\check{X}(t)$ *is not an approximate expansion of* $X(t)$.

5. The absolute convergence of the Fourier series

T. Kawata proved a theorem on a sufficient condition of the absolute convergence of the Fourier series of $x(t)$-process which is a probabilistic analogue of the Bernstein's theorem. We shall prove here a theorem on the generalized absolute convergence of the Fourier series of $X(t)$-process, which is an analogue of the theorem in [15] Page 137.

THEOREM 5-1. *If* $X(t)$ *is periodic with period* T, *and*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\lambda|^{\alpha}|d^2F(\lambda,\lambda')|<\infty, \qquad 0\leq\alpha\leq 2,$$

then
$$\sum_{n=-\infty}^{\infty}|C_n|^p$$

converges with probability one when $p>\frac{2}{\alpha+1}$,

(*Proof*)

$$\sum_{n=1}^{\infty}E|C_n|^p=\sum_{\nu=1}^{\infty}\sum_{n=2^{\nu-1}+1}^{2^\nu}E|C_n|^p\leq\sum_{\nu=1}^{\infty}\Big(\sum_{n=2^{\nu-1}+1}^{2^\nu}E|C_n|^2\Big)^{p/2}\Big(\sum_{n=2^{\nu-1}+1}^{2^\nu}1\Big)^{1-\frac{p}{2}}$$
$$\leq\sum_{\nu=1}^{\infty}O(2^\nu)^{-\alpha\cdot\frac{p}{2}}O(2^\nu)^{1-\frac{p}{2}} \qquad (by\ Thm.\ 2\text{-}1\ (ii))$$
$$=\sum_{\nu=1}^{\infty}O(2^{\frac{\nu}{2}(2-p(\alpha+1))})=O(1),$$

when $\frac{2}{\alpha+1}<p$.

Similarly we have

$$\sum_{n=-\infty}^{0}E|C_n|^p=O(1),$$

when $\frac{2}{\alpha+1}<p$.

Therefore by the Fubini theorem $\sum_{n=-\infty}^{\infty}|C_n|^p$ converges with probability one. *q.e.d.*

REMARK 5-1. *When $p=1$, the above theorem turns out to deal with the absolute convergence, i.e.*

COROLLARY 5-1 *If $X(t)$ is a process with period T and*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\lambda|^{\alpha}|d^2F(\lambda,\lambda')|<\infty \text{ for some } \alpha>1, \tag{5.1}$$

then $\sum_{n=-\infty}^{\infty}|C_n|$ converges with probability one.

Indeed, from the Theorem 5-1, we have now $\frac{2}{\alpha+1}>1$, i.e. $\alpha>1$.

More generally we have the following

REMARK 5-2. *The condition*(5.1) *can be replaced by the following condition:*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\lambda|(\log^{+}|\lambda|)^{\beta}|d^2F(\lambda,\lambda')|<\infty \text{ for some } \beta>2. \tag{5.2}$$ *(See* [8] *page* 27).

REMARK 5-3. *The following theorem is an analogue of the Szasz's theorem. (See* [1] *page* 156, *and* [8] *page* 28.)

If we put

$$\varphi(h)=4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin\frac{T}{2}(\lambda-\lambda')}{\frac{T}{2}(\lambda-\lambda')}\sin\frac{h}{2}\lambda\sin\frac{h}{2}\lambda' d^2F(\lambda,\lambda'),$$

$$\phi(h)=\frac{1}{h}\int_{0}^{h}\varphi(h)dh \qquad (h>0)$$

$$=2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin\frac{T}{2}(\lambda-\lambda')}{\frac{T}{2}(\lambda-\lambda')}\left(\frac{\sin\frac{h}{2}(\lambda-\lambda')}{\frac{h}{2}(\lambda-\lambda')}-\frac{\sin\frac{h}{2}(\lambda+\lambda')}{\frac{h}{2}(\lambda+\lambda')}\right)d^2F(\lambda,\lambda'),$$

then we have the following

THEOREM 5-2. *If $X(t)$ is periodic, with period T, and*

$$\sum_{n=1}^{\infty}\left(\phi\left(\frac{1}{n}\right)\right)^{1/2}n^{-1/2}<\infty,$$

then $\sum_{n=-\infty}^{\infty}|C_n|<\infty$ with probability one.

REMARK 5-4. *The proof of the above theorem is just the same as in the case of the Szasz's theorem as T. Kawata* [8] *remarks. And there is no difference between the case of the stationary process and the case of the harmonizable process. For the sake of completeness we shall prove it now. The following proof is adapted from N Bary* [1].

(*Proof*) From the Parseval relation we have

$$\varphi(h)=E\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\left|X\left(t+\frac{h}{2}\right)-X\left(t-\frac{h}{2}\right)\right|^2dt=4\sum_{k=-\infty}^{\infty}E|C_k|^2\sin^2\frac{\pi kh}{T}.$$

$$\int_{0}^{T/n}\varphi(h)dh=4\sum_{k=-\infty}^{\infty}E|C_k|^2\int_{0}^{T/n}\sin^2\frac{\pi kh}{T}dh$$

$$=4\sum_{k=-\infty}^{\infty}E|C_k|^2\frac{T}{\pi k}\int_{0}^{k\frac{\pi}{n}}\sin^2 t\,dt.$$

If l is the integer such that

$$l \leq \frac{h}{n} < l+1,$$

then

$$\frac{T}{\pi k}\int_0^{k\frac{\pi}{n}} \sin^2 t dt \geq \frac{T}{\pi n(l+1)}\int_0^{l\pi} \sin^2 t\, dt = \frac{Tl}{\pi n(l+1)}\int_0^{\pi} \sin^2 t\, dt = \frac{Tl}{n(l+1)}\cdot\frac{1}{2}.$$

If $k \geq n$ this last term is not less than $\frac{T}{n}\cdot\frac{1}{4}$.

Therefore

$$\frac{1}{T/n}\int_0^{\pi/n} \varphi(h)dh \geq \sum_{|k|\geq n} E|C_k|^2,$$

i.e.

$$\phi\left(\frac{T}{n}\right) \geq \sum_{|k|\geq n} E|C_k|^2.$$

As we can find a finite and positive constant B such that $B\phi\left(\frac{1}{n}\right) \geq \phi\left(\frac{T}{n}\right)$, we have

$$B\phi\left(\frac{1}{n}\right) \geq \sum_{|k|\geq n} E|C_k|^2.$$

Now

$$\begin{aligned}\sum_{k=-\infty}^{\infty} E|C_k| &= E|C_0| + \sum_{-k=1}^{\infty}\sum_{n=1}^{|k|}\frac{E|C_k|}{|k|} + \sum_{k=1}^{\infty}\sum_{n=1}^{k}\frac{E|C_k|}{k} \\ &= E|C_0| + \sum_{n=1}^{\infty}\sum_{-k=n}^{\infty}\frac{E|C_k|}{k} + \sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\frac{E|C_k|}{k} \\ &= E|C_0| + \sum_{n=1}^{\infty}\sum_{|k|\geq n}\frac{E|C_k|}{k} \\ &\leq E|C_0| + \sum_{n=1}^{\infty}\sqrt{\sum_{|k|\geq n}\frac{1}{k^2}\sum_{|k|\geq n} E|C_k|^2} \\ &\geq E|C_0| + \sum_{n=1}^{\infty}\sqrt{A\cdot\frac{1}{n}\cdot\phi\left(\frac{1}{n}\right)}.\end{aligned}$$

where A is a finite and positive constant.

6. Covariances of Fourier coefficients

The covariance $EC_n\bar{C}_m$ of Fourier coefficients can be written as follows.

$$(6.1)\qquad EC_n\bar{C}_m = \frac{1}{T^2}\int_{-\frac{T}{2}}^{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\frac{2\pi}{T}(nt-mt')}R(t,t')dtdt',$$

or

$$(6.2)\qquad EC_n\bar{C}_m = (-1)^{n+m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin\frac{T\lambda}{2}}{\frac{T\lambda}{2}}\cdot\frac{\sin\frac{T\lambda'}{2}}{\frac{T\lambda'}{2}}\cdot\frac{1}{1-\frac{2\pi n}{T\lambda}}\cdot\frac{1}{1-\frac{2\pi m}{T\lambda'}}d^2F(\lambda,\ \lambda').$$

Properties of coefficients $Ec_n\bar{c}_m$ was investigated by T. Kawata [7] and W. L. Root & T.S.Pitcher [14]. The former approached it from the conditions on the spectral distribution function, while the latter from those on the covariance $\rho(u)$ itself.

We shall give here one theorem, starting from the relation (6.1), which is an analogue of one of theorems in [14].

THEOREM6-1 *If* $\int_{-\infty}^{\infty}|R(t,\ t+u)|\ du \leq K$ *uniformly for* t, *for some constant* K, *if there exists* $\psi(v)\in L_1\left(-\frac{1}{2},\ \frac{1}{2}\right)$ *such that* $\left|\int_{-T\left(\frac{1}{2}+v\right)}^{T\left(\frac{1}{2}-v\right)} R(Tv, Tv+u)du\right| \leq \psi(v)$, *and if*

$$\lim_{T\to\infty}\int_{-T\left(\frac{1}{2}+v\right)}^{T\left(\frac{1}{2}-v\right)} R(Tv, Tv+u)du=\varphi(v) \quad \text{exists,}$$

then

$$TEC_n\bar{C}_m \to \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i(n-m)v}\varphi(v)dv.$$

If moreover $\varphi(v)$ *happens to be independent of* v *and non-zero, then whenever* $n\neq m$, *we have*

$$\lim_{T\to\infty} EC_n\bar{C}_m/\sqrt{E|C_n|^2E|C_m|^2}=0.$$

(*Proof*)

$$TEC_n\bar{C}_m=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-\frac{2\pi i}{T}(nt-mt')} R(t,t')dtdt'.$$

By the transformation $t'-t=u$, we have

$$TEC_n\bar{C}_m=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-\frac{2\pi i}{T}(n-m)t}dt\int_{-\frac{T}{2}-t}^{\frac{T}{2}-t} e^{\frac{2\pi imu}{T}} R(t,t+u)du.$$

By the transformation $t/T=v$, we have

$$TEC_n\bar{C}_m=\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i(n-m)v}dv \int_{-\frac{T}{2}-Tv}^{\frac{T}{2}-Tv} e^{\frac{2\pi imu}{T}} R(Tv, Tv+u)du.$$

As we may think of the case only when $|v|<1/2$, we have

$$\begin{aligned}
&\int_{-\frac{T}{2}-Tv}^{\frac{T}{2}-Tv} e^{\frac{2\pi imu}{T}} R(Tv, Tv+u)du \\
&=\int_{-T\left(\frac{1}{2}+v\right)}^{T\left(\frac{1}{2}+v\right)} R(Tv, Tv+u)du \\
&\quad+\int_{-T\left(\frac{1}{2}+v\right)}^{T\left(\frac{1}{2}-v\right)} \left(e^{\frac{2\pi imu}{T}}-1\right)R(Tv, Tv+u)du \\
&=J_1+J_2,
\end{aligned}$$

for any large positive number A, there exists a number T so large that

$$T\left(\frac{1}{2}-v\right)>A,\ \ T\left(\frac{1}{2}+v\right)>A,\ \ J_2=\int_{-A}^{A}+\int_{A}^{T\left(\frac{1}{2}-v\right)}+\int_{-T\left(\frac{1}{2}+v\right)}^{-A}.$$

$$\left|\int_{A}^{T\left(\frac{1}{2}-v\right)}\left(e^{\frac{2\pi imu}{T}}-1\right)R(Tv, Tv+u)du\right|$$

$$\leq 2\int_A^{T(\frac{1}{2}-v)} |R(Tv, Tv+u)|du$$
$$\leq 2\int_A^{\infty} |R(Tv, Tv+u)|du < 2\varepsilon,$$
$$\left|\int_{-A}^{A}\left(e^{\frac{2\pi mui}{T}}-1\right)R(Tv,\ Tv+u)du\right|$$
$$\leq \varepsilon\int_{-A}^{A} |R(Tv, Tv+u)|du \leq \varepsilon K,$$

where we used the fact that when T is large enough, for $|u|\leq A$, $|e^{\frac{2m\pi ui}{T}}-1|<\varepsilon$ for any $\varepsilon>0$.

Therefore $|J_2|<\varepsilon C$ for some constant C, i.e. $\lim_{T\to\infty} J_2=0$ for each $|v|<1/2$

Hence
$$\lim_{T\to\infty}\int_{-\frac{T}{2}-Tv}^{\frac{T}{2}-Tv} e^{\frac{2\pi imu}{T}} R(Tv, Tv+u)du$$
$$=\lim_{T\to\infty}\int_{-\frac{T}{2}-Tv}^{\frac{T}{2}-Tv} R(Tv,\ Tv+u)du.$$

$$\lim_{T\to\infty} TEC_n\bar{C}_m=\lim_{T\to\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i(n-m)v}dv\int_{-T(\frac{1}{2}+v)}^{T(\frac{1}{2}-v)} R(Tv,\ Tv+u)du$$
$$=\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i(n-m)v}dv\lim_{T\to\infty}\int_{-T(\frac{1}{2}+v)}^{T(\frac{1}{2}-v)} R(Tv,\ Tv+u)du$$
$$=\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i(n-m)v}\varphi(v)dv.$$

When $\varphi(v)$ happens to be a positive constant
$$\lim_{T\to\infty} TEC_n\bar{C}_m=a\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i(n-m)v}dv$$
$$=\begin{cases}0 \text{ if } n\neq m,\\ a \text{ if } n=m.\end{cases}$$
q.e.d.

REMARK 6-1 *As we did not use in the above proof the representation* (6.2), *our theorem holds for more general non-stationary processes including the harmonizable process.*

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