A Note on a Finite Difference Analogue of Mixed Boundary Value Problem

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1. Introduction. In this paper we are concerned with a finite difference approximation to the solution of the boundary value problem

$$\begin{aligned} Lu = -(au_{xx}+2bu_{xy}+cu_{yy}) = f \quad & \text{in } R \\ \frac{\partial u}{\partial n}+\alpha u = g \quad & \text{on } C_1 \\ u = h \quad & \text{on } C_2 \end{aligned} \tag{1.1}$$

The region R is a bounded connected open set in the (x, y) plane whose boundary C consists of two parts C_1 and C_2. The symbol $\frac{\partial}{\partial n}$ denotes differentiation with respect to outward directed normal on C_1. The function f, g and h are defined to be sufficiently smooth functions on R, C_1 and C_2 respectively. The numbers a, b and c are constants such that $b^2-ac<0$, and $a>0$, namely the operator L is uniformly elliptic. The boundaries C_1 and C_2 are unions of finite number of arc elements. The function α is a piecewise differentiable function on C_1. We restrict α to be zero on $C_1^{(1)}$ and positive on $C_1^{(2)}$, $C_1^{(1)}$, $C_1^{(2)}$ being portions of C_1 for which $C_1=C_1^{(1)}\cup C_1^{(2)}$, $C_1^{(1)}\cap C_1^{(2)}\equiv\phi$. We also assume that $C_1\not\equiv\phi$, and that if $C_2\equiv\phi$, then there is an arc element of non-zero measure in $C_1^{(2)}$. We shall present some well known results from matrix theory which is used in the following proofs.

A matrix A is said to be *non-negative* if each element of A is non-negative and the notation $A\geqq 0$ will be used.

A matrix B with elements b_{ij} is said to be *monotone* if $x\geqq 0$ for any vector x such that $Bx\geqq 0$.

A characterization of monotone matrices is given by the following theorem, cf. [1], [2].

THEOREM. *The matrix B is monotone if and only if B is nonsingular and $B^{-1}\geqq 0$.*

DEFINITION. An $n\times n$ matrix B with elements b_{ij} is said to be *of positive type* if the followig conditions are satisfied:

a) $b_{ji}\leqq 0 \quad i\neq j$

b) $\sum_k b_{jk}\geqq 0$ for all j, and further there exists a non-empty subset $J(B)$ of the integers $1, 2, \cdots, n$ such that for all $j\in J(B)$, $\sum_k b_{jk}>0$

c) for $i\notin J(B)$ there exists a $j\in J(B)$ and a sequence of non-zero elements of B which is of the form

$$b_{ik_1},\ b_{k_1k^2},\ \cdots,\ b_{k_rj}$$

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THEOREM. *If B is of positive type then B is monotone. cf.* [4], [1].

2. Finite-difference analogue. We shall first transform Lu into a form which lends itself to the formulation of a finite difference problem of non-negative type (at points of R). We define the new coordinates (ξ, η) resulting from the rotation

$$\xi = x \cos\theta - y \sin\theta,$$
$$\eta = x \sin\theta + y \cos\theta.$$

Setting $u(x,y) = v(\xi, \eta)$ and choosing θ in such a way that $(c-a)\tan 2\theta = 2b$ (if $a \neq c$), $\cos 2\theta = 0$ (if $a = c$), we see that the operator Lu is expressed as follows

$$Lu = -(Av_{\xi\xi} + Bv_{\eta\eta}),$$

where $A > 0$, $B > 0$.

We place a square mesh of width h on the region R with respect to (ξ, η) axis, and call the mesh crossings "mesh points". The set R_h will consists of those mesh points of R whose four nearest neighbors are in R. The intersection of the mesh with C_i will make up the set C_{ih}, $i=1,2$. The set C^*_{ih} will denote those mesh points of R which are at a distance less than or equal to h (along the (ξ, η) axis from C_{ih}, $i=1,2$).

We define the following operators at a point (ξ, η) of R_h.

$$L_h v(\xi,\eta) = -h^{-2}\{Av(\xi+h,\eta) + Av(\xi-h,\eta) + Bv(\xi,\eta+h) + Bv(\xi,\eta-h) - 2(A+B)v(\xi,\eta)\}.$$

It is well known that for $u \in C^{(3)}(\bar{R})$

$$|Lu(P) - L_h u(P)| \leq M_1 h, \quad P \in R_h \tag{2.1}$$

where M_1 is a constant depending on the third derivative of u. cf. [3]. On C^*_{ih} we use the following operator.

$$L_h v(\xi,\eta) = -h^{-2}\Big\{\frac{A}{\lambda(1+\lambda)}v(\xi-\varepsilon_1\lambda h,\eta) + \frac{A}{1+\lambda}v(\xi+\varepsilon_1 h,\eta) + \frac{B}{\mu(1+\mu)}v(\xi,\eta-\varepsilon_2\mu h) + \frac{B}{1+\mu}v(\xi,\eta+\varepsilon_2 h) - \Big[\frac{A}{\lambda} + \frac{B}{\mu}\Big]v(\xi,\eta)\Big\}$$

where

$$\varepsilon_1 = \pm 1, \quad \varepsilon_2 = \pm 1, \quad 0 < \lambda, \mu \leq 1.$$

For example, if (ξ, η) is the point (ξ, η) in Figure 1, then the inequality

$$|Lu(P) - L_h u(P)| \leq M_2 h, \quad P \in C^*_{1h} + C^*_{2h}, \tag{2.2}$$

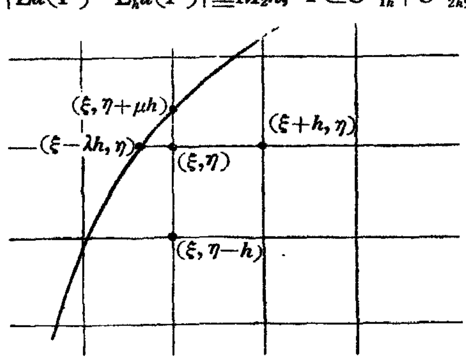

Figure 1

where M_2 depends on the third derivative of u, is easily verified. For the normal operator we adopt Greenspan's method. [5]

Case A. Suppose that at $(x,y)\in C_{1h}$, numbered 0 (see Figure 2, a), the axis of the normal is directed inward and first meets the lattice in a point of $R_h+C^*_{ih}$ which has been numbered 1. We define the operator δ_n to be

$$\delta_n u_0=\frac{u_0-u_1}{d}.$$

Case B. Suppose that at $(x,y)\in C_{1h}$, numbered 0 (see Figure 2, b) the axis of the normal is directed inward and first meets with the lattice in a point numbered 1 that is not a point of $R_h+C^*_{ih}$. Then this point lies in the interior of a closed segment of the lattice which contains exactly two points, numbered 2 and 3, of $R_h+C^*_{ih}+C_{ih}$ and of which, we assume, at least one is a point of $R_h+C^*_{ih}$. The normal operator δ_n in this case is defined by

$$\delta_n u_0=\frac{u_0}{d}-\frac{d_2}{d(d_2+d_3)}u_3-\frac{d_3}{d(d_2+d_3)}u_2.$$

Case C. Suppose that at $(x,y)\in C_{1h}$, and numbered 0 (see Figure 2, c), the normal is parallel to a line of the lattice. Suppose then that the associated axis is directed inward and meets with a first lattice point that is a point of C^*_{ih} and is numbered 1. Then set

$$\delta_n u_0=\frac{u_0-u_1}{d}.$$

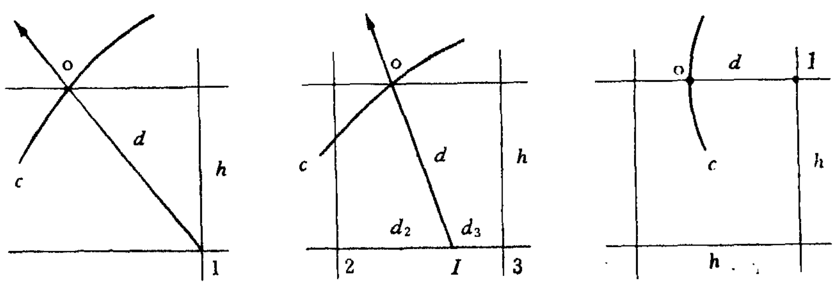

Figure 2

In any case of above the inequality

$$\left|\frac{\partial u(P)}{\partial n}+\alpha(P)u(P)-[\delta_n u(P)+\alpha(P)u(P)]\right|\leq M_3 h,\quad P\in C_{1h} \tag{2.3}$$

where M_3 depends on the second derivatives of u, is easily verified.

We now pose the following finite-difference analogue of (1.1)

$$\begin{aligned} L_h u(P)&=f(P),\quad P\in R_h+C^*_{1h}+C^*_{2h}\\ u(P)+\alpha(P)u(P)&=g(P),\quad P\in C_{1h}\\ u(P)&=h(P),\quad P\in C_{2h}. \end{aligned}\tag{2.4}$$

Since $\alpha(P)\geqq 0$, the matrix of the system (2.4) is of positive type. Thus the system (2.4) has the unique solution and we may introduce Green's function corresponding to

(2.4).

Let $G_h(P,Q)$ be defined as follows:

$$L_{h,p}G_h(P,Q)=h^{-2}\delta(P,Q), \qquad P\in R_h+C^*_{1h}+C^*_{2h}$$
$$\delta_n G_h(P,Q)+\alpha(P)G_h(P,Q)=\delta(P,Q), \quad P\in C_{1h}$$
$$G_h(P,Q)=\delta(P,Q), \qquad P\in C_{2h}$$

for $Q\in R_h+C^*_{1h}+C^*_{2h}+C_{1h}+C_{2h}$.

Since the matrix of the system (2.4) is of positive type, it follows that $G_h(P,Q)\geqq 0$.

LEMMA 1. *Let $V(P)$ be an arbitrary mesh function defined on $R_h+C^*_{1h}+C^*_{2h}+C_{1h}+C_{2h}$. Then*

$$\begin{aligned} V(P)= &\sum_{Q\in R_h+C_{1h}+C_{2h}} G_h(P,Q)[L_h V(Q)] \\ &+\sum_{Q\in C_{1h}} G_h(P,Q)[\delta_h V(Q)+\alpha(Q)V(Q)] \\ &+\sum_{Q\in C_{2h}} G_h(P,Q)V(Q). \end{aligned} \tag{2.5}$$

Proof. Let $W(P)$ denote the right hand side of (2.5). Then

$$L_h W(P)=L_h V(P), \qquad P\in R_h+C^*_{1h}+C^*_{2h}$$
$$\delta_n W(P)+\alpha(P)W(P)=\delta_n V(P)+\alpha(P)V(P), \quad P\in C_{1h}$$
$$W(P)=V(P) \qquad P\in C_{2h}$$

From the uniqueness of solution of (2.4), we have

$$W(P)=V(P), \quad P\in R_h+C^*_{1h}+C^*_{2h}+C_{1h}+C_{2h}.$$

Letting $V(P)=1$ in (2.5), we have

$$\sum_{Q\in C_{2h}} G_h(P,Q)\leqq 1.$$

Now suppose R is such that there exists a function $\phi\in C^{(3)}(\bar{R})$ satisfying

$$L\phi\geqq 1 \text{ in } R$$
$$\frac{\partial\phi}{\partial n}+\alpha\phi\geqq 1 \text{ on } C_1. \tag{2.6}$$

LEMMA 2. *Suppose that the function ϕ of (2.6) exists. Then for small enough h*

$$\sum_{Q\in C_{2h}} G_h(P,Q)+h^2\sum_{Q\in R+C_{1h}+C_{2h}} G_h(P,Q)\leqq 4\,[\max_{Q\in\bar{R}}|\phi(Q)|] \tag{2.7}$$

Proof. For small enough h,

$$L_h\phi(P)\geqq\frac{1}{2}, \qquad P\in R_h+C^*_{1h}+C^*_{2h}$$

and

$$\delta_n\phi(P)+\alpha(P)\phi(P)\geqq\frac{1}{2}, \qquad P\in C_{1h}.$$

Taking $V(P)=\phi(P)$ in (2.5), we have

$$\phi(P)\geqq h^2\sum_{Q\in R_h+C^*_{1h}+C^*_{2h}}\frac{1}{2}G_h(P,Q)+\sum_{Q\in C_{1h}}\frac{1}{2}G_h(P,Q)+\sum_{Q\in C_{2h}}G_h(P,Q)\ \phi(Q)$$

Sinnce $\sum_{Q\in C_{2h}} G_h(P,Q)\leqq 1$, we have the inequality (2.7).

We are now in a position to prove the following theorem.

THEOREM: *Let $u\in C^{(3)}(\bar{R})$ be the solution of (1.1). Suppose that the function ϕ of (2.6) exists. Then*

$$\varepsilon(P) \equiv u(P) - U(P), \quad P \in R_h + C^*_{1h} + C^*_{2h} + C_{1h} + C_{2h},$$

where $U(P)$ *is the solution of* (2.4), *satisfies the inequality*

$$\max_P |\varepsilon(P)| \leqq kh, \tag{2.8}$$

where k *is a constant which depends on* u *and* ϕ *but not on* h.

Proof. Let $V(P) = \varepsilon(P)$ in (2.5). Then

$$\varepsilon(P) = h^2 \sum_{Q \in R_h + C^*_{1h} + C^*_{2h}} G_h(P, Q)[L_h \varepsilon(Q)]$$
$$+ \sum_{Q \in C_{1h}} G_h(P, Q) \; [\delta_n \varepsilon(Q) + \alpha(Q)\varepsilon(Q)].$$

Since $G_h(P, Q) \geqq 0$, we have

$$|\varepsilon(P)| \leqq [h^2 \sum_{Q \in R_h + C^*_{1h} + C^*_{2h}} G_h(P, Q)] \cdot \max_{Q \in R_h + C^*_{1h} + C^*_{2h}} |L_h \varepsilon(Q)| \tag{2.9}$$
$$+ [\sum_{Q \in C_{1h}} G_h(P, Q)] \cdot \max_{Q \in C_{1h}} |\delta_n \varepsilon(Q) + \alpha(Q)\varepsilon(Q)|$$

From (2.1), (2.2) and (2.3) we have

$$|L_h \varepsilon(P)| \leqq Mh, \qquad P \in R_h + C^*_{1h} + C^*_{2h} \tag{2.10}$$
$$|\delta_n \varepsilon(P) + \alpha(P)\varepsilon(P)| \leqq M_3 h, \quad P \in C_{1h},$$

where $M = \max \{M_1, M_2\}$. Thus the theorem follows by inserting (2.7) and (2.10) into (2.9).

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