On Weakly-Primitive Ideals

JAE K. PARK

1. **Introduction.** Let R_n be the ring of all $n\times n$ matrices over a given ring R, where n is a positive integer.

In [5], it was shown that a ring R is weakly-primitive if and only if R_n is weakly-primitive.

In this paper the (right) weakly-primitive ideal will be defined and using the result of [5], the author will show that an ideal Q of R_n is (right) weakly-primitive if and only if $Q=P_n$ for some (right) weakly-primitive ideal P of R, which is analogous to a result in [7].

In section 2, some terminologies and lemmas will be introduced. And in section 3, the main results of this paper which is mentioned above will be shown.

2. **Preliminaries.** A right ideal I of a ring R is called *irreducible* [3, p. 63] provided that the (right) R-module $R-I$ is a *uniform* R-module, that is, if J_1 and J_2 are right ideals of R such that $J_1\cap J_2=I$, then either $J_1=I$ or $J_2=I$. The *normalizer* $N(I)$ of I is the set of all elements r of R such that $rI\subseteq I$. Clearly $N(I)$ is the largest subring of R containing I as an ideal of $N(I)$. For each a in R, let $[I:a]$ denote the set of elements r of R satisfying $ar\in I$.

We recall that a proper right ideal I of R is called *almost-maximal* [3, p. 63] if I satisfies the following conditions:

(1) I is irreducible.

(2) If $[I:a]\supset I$ for a in R, then $a\in I$.

(3) For any right ideal $J\supset I$, we have $N(I)\cap J$ properly contains I. Moreover, if $[J:a]\supseteq I$, then $[J:a]\supset I$.

Following [5, p. 7], a ring R is called *weakly-primitive* if R has an almost-maximal right ideal J such that $[J:R]=\{r\in R\mid Rr\subseteq J\}=(0)$.

Now we shall define the weak-primitivity of ideals.

DEFINITION. Let R be a ring and I be a (two-sided) ideal of R. Then I is called (right) *weakly-primitive* if R/I is a (right) weak-primitive ring.

Although one can find the following lemma in [2], the proof will be given here for completion.

LEMMA 2.1. *Let Q be an ideal of a ring R, and let I be a right ideal of R such that $R\supset I\supseteq Q$. Then I/Q is almost-maximal in R/Q if and only if I is almost-maximal in R.*

Proof. Suppose that I/Q is almost-maximal. Let J_1 and J_2 be right ideals of R with $J_1\cap J_2=I$, then we have $J_1/Q\cap J_2/Q=I/Q$. Since I/Q is irreducible, either $J_1/Q=I/Q$ or $J_2/Q=I/Q$. Therefore, $J_1=I$ or $J_2=I$ and the irreducibility of I follows. Now assume $[I:a]\supset I$, $a\in R$, then there exists r in R such that $r\notin I$, $ar\in I$. Let $\bar{r}=r+Q$, then $\bar{r}$

Received by the editors March 20, 1970.

does not belong to $[I/Q:a+Q]$ because $[I/Q:a+Q]=[I:a]/Q$. Since $\bar{r}$ does not belong to I/Q, it is true that $[I/Q:a+Q]/Q$ properly contains I/Q. It follows that $a+Q\in I/Q$, that is, $a\in I$ by the almost-maximality of I/Q. Now let J be a right ideal of R such that $J\supset I$. Clearly, we have $J/Q\supset I/Q$ and $N(I/Q)\cap J/Q\supset I/Q$. And straightfowardly $N(I/Q)=N(I)/Q$. From the above, we have $N(I/Q)\cap J/Q=N(I)/Q\cap J/Q=(N(I)\cap J)/Q\supset I/Q$, i.e. $N(I)\cap J$ properly contains I. Furthermore assume that $J\supset I$ and $[J:a]\supseteq I$. It is clear that $[J:a]/Q\supset I/Q$ and $J/Q\supset I/Q$ hold. Since $[J:a]/Q=[J/Q:a+Q]$, we have $[J/Q:a+Q]\supseteq I/Q$. Almost maximality of I/Q in R/Q implies $[J/Q:a+Q]=[J:a]/Q \supset I/Q$. Therefore $[J: a]\supset I$. The converse of this Lemma can be proved similarly.

Using this, we have the following result which is an analogue of Jacobson [1, p.7].

LEMMA 2.2. *An ideal Q of a ring R is weakly-primitive if and only if there exists an almost-maximal right ideal J of R such that $[J:R]=Q$.*

Proof. By the definition of a weakly-primitive ideal, if Q is weakly-primitive then so is R/Q that is, there exists an almost maximal right ideal J/Q of R/Q such that $[J/Q:R/Q]=(\bar{0})$. Since $[J/Q:R/Q]=[J:R]/Q=(\bar{0})$, we have $[J:R]=Q$. And the almost-maximality of J follows from Lemma 2.1.

Conversely suppose that there exists an almost maximal right ideal J in R such that $[J:R]=Q$, then $[J/Q:R/Q]=(\bar{0})$. And again by Lemma 2.1, J/Q is almost-maximal. This means that the ideal Q is almost-maximal. This completes the proof.

From Lemma 2.2, it follows that the intersection of the (right) weakly-primitive ideals of a ring R is the *right weak radical* $W_r(R)$ defined in [4, p.554].

REMARK. From [3, p.64, corollary 2.2] in any ring if Q is a weakly-primitive ideal, then Q is a prime ideal. And clearly a primitive ideal is weakly-primitive. From these it follows that $B(R)\subseteq W_r(R)\subseteq J(R)$, where $B(R)$, $J(R)$ denotes prime radical, the Jacobson radical respectively. In particular if a ring R is right-Artinian, then $B(R)$, $W_r(R)$ and $J(R)$ are coincided [6, p.120, corollary 6.26] and [1, p.39, corollary 2]. In this case the right weak radical $W_r(R)$ and left weak radical $W_l(R)$ coincide.

Now the following lemma can be proved straightfowardly, so the proof will be omitted.

LEMMA 2.3. *Let Q be an ideal of R, then $(R/Q)_n=R_n/Q_n$.*

3. The existence of the weakly-primitive ideal. Now we are in a position to prove the following main theorems:

THEOREM 3.1. *If P is a (right) weakly-primitive ideal of R, then P_n is also a (right) weakly-primitive ideal of R_n.*

Proof. If P is a (right) weakly-primitive ideal, then R/P is a (right) weakly-primitive ring. By [5, Theorem 1] and lemma 2.3, $(R/P)_n=R_n/P_n$ is weakly-primitive. Therefore P_n is a (right) weakly-pimitive ideal of R_n.

The following theorem is the convese of Theorem 3.1.

THEOREM 3.2. *If Q is a (right) weakly-primitive ideal of R_n there exists a (right) weakly-primitive ideal P of R such that $Q=P_n$.*

Proof. If Q is a (right) weakly-primitive ideal of R_n, then by Lemma 2.2, there exists an almost-maximal right ideal J^* of R_n such that $[J^*: R_n]$ equals to Q. Now let P be the set of all entries of the elements of Q. Clearly $Q \subseteq P_n$ holds. To show the converse inclusion $P_n \subseteq Q$ hold, it is sufficient to prove that $aE_{pq} \in Q$ for any a in P, and any p, q; $1 \leqq p, q \leqq n$, where aE_{pq} denotes the matrix with a in (p, q)-position and elsewhere 0. Suppose $aE_{pq} \notin Q$ for some $a \in P$ and p, q; $1 \leqq p, q \leqq n$. Then there exists $\rho = (r_{ij})$ in R_n such that $\rho(aE_{pq})$ does not belong to J^*. Now since

$$\rho(aE_{pq}) = \sum_{i=1}^{n} r_{ip} aE_{iq} \notin J^*,$$

there exists k; $1 \leqq k \leqq n$ such that

$$r_{kp} aE_{kp} \notin J^*.$$

In this case by [5, corollary of Lemma 1], there exists $\sigma = (s_{ij}) \in R_n$ such that

$$(r_{kp} aE_{kq})(s_{ij}) \notin J^*$$

and

$$(r_{kp} aE_{kq})(S_{ij}) \in N(J^*).$$

Since

$$(r_{kp} aE_{kq})(s_{ij}) = \sum_{j=1}^{n} r_{kp} as_{qj} E_{kj} \notin J^*,$$

for some j; $1 \leqq j \leqq n$, it follows that

$$r_{kp} as_{qj} E_{kj} \notin J^*.$$

For the simplicity, we write $r = r_{kp}$, $s = s_{qj}$. From the above, if $aE_{pq} \notin Q$ for some $a \in P$ and for some p, q; $1 \leqq p, q \leqq n$, then there exist r, s in R such that

$$rasE_{kj} \notin J^*$$

for some k, j; $1 \leqq k, j \leqq n$. Since $a \in P$, there exists $\alpha = (a_{ij}) \in Q$ such that $a_{lm} = a$ for some l, m; $1 \leqq l, m \leqq n$. And now since

$$rasE_{kj} = (rE_{kl})\alpha(sE_{mj}), \quad \alpha \in Q,$$

$rasE_{kj} \in Q$. On the other hand,

$$Q = [J^* : R_n] = \bigcap_{\rho \in R_n} [J^* : \rho] \subseteq \bigcap_{\rho \in N(J)^*, \rho \notin J^*} [J^*] : \rho].$$

If $\rho \in N(J^*)$, $\rho \notin J^*$, then $[J^* : \rho] = J^*$ by the almost-maximality of J^*. Therefore

$$Q \subseteq \bigcap_{\rho \in N(J^*), \rho \notin J^*} [J^* : \rho] = J^*.$$

Hence $rasE_{kj} = (rE_{kl})\alpha(sE_{nj}) \in Q \subseteq J^*$, which is a contradiction. Therefore $P_n = Q$. Now P is an ideal of R since $P_n = Q$ is an ideal of R_n. Now $(R/P)_n = R_n/P_n = R_n/Q$ is weakly-primitive, R/P is weakly-primitive by [5, Theorem 2]. This means P is a (right) weakly-primitive ideal of R. Now the proof is completed.

REMARK. The author wishes to thank Professor Wuhan Lee for his suggestions in the preparation of this paper.

References

[1] N. Jacobson, *Structure of rings*, 2nd rev. ed., Amer. Math. Soc., Providence, R.I., 1964.
[2] Kwangil Koh, *On almost maximal right ideals*, Proc. Amer. Math. Soc. **25** (1970), 266-272.
[3] Kwangil Koh and A.C. Mewborn, *A class of prime ring*, Canad. Math. Bull. (1), **9** (1966),

63-72.

[4] ______, *The Weak radical of a ring*, Proc. Amer. Math. Soc., 18 (1967), 554-559.

[5] Wuhan Lee, *On the weak-primitivity of matrix rings*, J. Korean Math. Soc., 7 (1970), 7-14.

[6] Neal H. McCoy, *The theory of rings*, Macmillan Company, 1964.

[7] E.C. Posner, *Primitive matrix rings*, Archiv Math., 12 (1961), 97-101.

Seoul National University