Inf-Preserving Functors from $\underline{A}$ to Ens

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1. Introduction. Let I and $\underline{A}$ be an index category and a small category respectively. For each object A of $\underline{A}$ the constant diagram $A_I: I \to \underline{A}$ is defined by $A_I(i)=A$, $A_I(k)=1_A$ for each object i and map k: $i \to j$ of I. A lower bound (A,u) of a diagram Γ: $I \to \underline{A}$ consists of an object A of $\underline{A}$ and a natural transformation u: $A_I \to \Gamma$. The lower bound (A,u) of Γ will be called the infimum of Γ if for every lower bound (A',u') of Γ there exists a unique map a: $A' \to A$ such that $u(i)a=u'(i)$ for all objects i of I, and we write it by $\inf \Gamma \simeq (A,u)$. A functor F: $\underline{A} \to \underline{B}$ is called *the inf-preserving functor* or we say that it *preserves the infimums* if for every diagram Γ: $I \to \underline{A}$, $\inf \Gamma \simeq (A,u) \Rightarrow \inf (F,\Gamma)=(F(A),\ F\cdot u)$. An upper bound, the supremum of a diagram and the sup-preserving functors are also defined dually. We write the opposite category of $\underline{A}$ and the category of sets by $\underline{A}^\circ$ and *Ens* respectively. J. Lambeck [1] proved that $\underline{A}$ is embedded as a sup-dense subcategory into a sup-complete category $\underline{A}'$ of all functors from $\underline{A}^\circ$ to *Ens* and the embedding functor of $\underline{A}$ into the category $\underline{A}''$ of all inf-preserving functors from $\underline{A}^\circ$ to *Ens* is sup-dense and sup-preserving. Further he proved that the category $\underline{A}''$ is inf-complete. The purpose of this note is to prove that the opposite $\underline{A}'''^\circ$ of the category $\underline{A}'''$ of all inf-preserving functors from $\underline{A}$ to *Ens* is sup-complete and it is inf-complete if for any diagram Θ with $\inf \Theta=(z,t)$, t is a natural equivalence.

Throughout this note we assume that every diagram has the small index category.

2. Inf-preserving functors. Let $\{o\}$ be a typical one element set and T: $\underline{A} \to Ens$. We may associate with the element x of $T(A)$ for all A of $\underline{A}$ the map $\hat{x}: \{o\} \to T(A)$ such that $\hat{x}(o)=x$. The following lemmas will be stated whose proofs are to be found in [1] and [3] respectively.

LEMMA 1. *For any object A of $\underline{A}$ the functors $[A,\ \ ]$: $\underline{A} \to Ens$ and $[\ \ ,A]$: $\underline{A}^\circ \to$ Ens preserves infimums.*

LEMMA 2. *Let T and T' be two functors from $\underline{A}$ to $\underline{B}$ and $\eta: T \to T'$ a natural equivalence. Then T preserves infimums if and only if T' does.*

PROPOSITION 1. *The functor T: $\underline{A} \to \underline{B}$ preserves infimums if and only if $[B,\ T(\ )]$: $\underline{A} \to$Ens preserves infimums for all B in $\underline{B}$.*

Proof. Assume that T preserves infimums. The functor $[B, T(\ )]$ arises by composition from the inf-preserving functor $T: \underline{A} \to \underline{B}$ and the functor $[B,\ \ ]$: $\underline{B} \to Ens$. But the functor $[B,\ \ ]$ preserves infimum by the lemma 1. Hence it also preserves infimums. Conversely, assume that the functor $[B, T(\ )]$ preserves infs for all B in $\underline{B}$. Let D: $I \to \underline{A}$ be a diagram in $\underline{A}$ with $\inf D=(A,u)$. Then the infimum of the diagram $[B, T(\ )]\cdot D$: $I \to$ *Ens* is $([B, T(A)], v)$, where $v(i)=[B, T(u(i))]$ for each i of I. Let $t(i)$: $B \to T(D(i))$ be natural in i of I. We associate with $t(i)$ mapping $\hat{t}(i)$: $\{o\} \to [B, T(D(i))]$. Hence there exists a unique map $g: \{o\} \to [B, T(A)]$ such that $t(i)=v(i)\cdot g$. Since $\hat{t}(i)(o)=v$

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$(i)\cdot(g(o))$, we have a unique element $g(o)$ of $[B, T(A)]$ such that $t(i)=v(i)g(o)=T(u(i))\cdot g(o)$. Hence $\inf TD=[T(A), T\cdot u]$.

PROPOSITION 2. *Let $T:\underline{A}\to\underline{C}$ be an embedding functor and $\underline{C}_0$ be the subcategory of $\underline{C}$, consisting of all objects C in $\underline{C}$ such that the functor $[C, T(\ )]: \underline{A}\to$ Ens preserves infimums. Then*

(i) *the image $T(A)$ of T is contained in $\underline{C}_0$.*

(ii) *$\underline{C}_0$ is the largest subcategory of $\underline{C}$ such that the induced embedding functor $\underline{A}\to T(\underline{A})\to\underline{C}_0$ preserves infimums.*

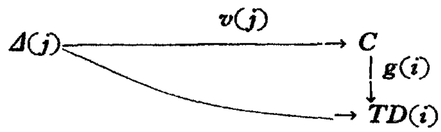
(iii) *Every supremum of any diagram Δ: $J\to\underline{C}$ in $\underline{C}$ is contained in $\underline{C}_0$.*

(iv) *For any diaram Θ: $K\to\underline{C}_0$ with $\inf \Theta=(C', t)$ in $\underline{C}$ if t is a natural equivalence then C' is contained in $\underline{C}_0$.*

Proof. (i) Since T is an embedding functor, we have a natural equivalence $\mu: [A,\ \ ]\cong[T(A), T(\ )]$ for each object A of $\underline{A}$, where $[A,\ \ ]$ and $[T(A), T(\ )]$ are two functors from $\underline{A}$ to *Ens*. The functor $[A,\ \ ]$ preserves infimums by the lemma 1. Hence the functor $[T(A), T(\ )]$ preserves infimums, by the lemma 2. Therefore $T(A)\in\underline{C}_0$ and $T(\underline{A})\subset\underline{C}_0$,

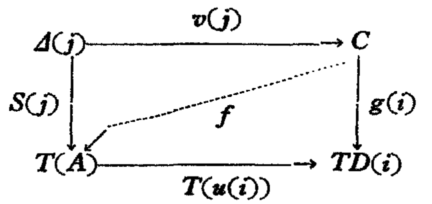
(ii) Let $\underline{A}\to T(\underline{A})\to\underline{C}'$ be an inf-preserving induced embedding functor. By the proposition 1, for each object C': of $\underline{C}'$: the functor $[C', T(\ )]$ preserves infimums. Hence C' is contained in $\underline{C}_0$.

(iii) Let Δ: $J\to\underline{C}_0$ be any diagram with $\sup \Delta=(C, v)$ in $\underline{C}$. Assume that D: $I\to\underline{A}$ is a diagram with $\inf D=(A, u)$ then $\inf T\cdot D=((TA), T\cdot u)$ in $\underline{C}_0$. Let g: $C_I\to TD$ be a natural transformation, then we have the map $g(i)v(j)$ in $\underline{C}$ such that



commutes.

Hence there exists a unique map $S(j): \Delta(j)\to T(A)$ such that



commutes.

Since $T(u(i))\cdot S(j)$: $\Delta(j)\to TD(i)$ is a natural in $i\in I$, so does $S(j)$ in $j\in J$. Hence there exists a unique map f: $C\to T(A)$ such that $S(j)=f\cdot v(j)$. Therefore $T(u(i))\cdot S(j)=T(u(i))f\cdot v(j)=g(i)v(j)$, hence $g(i)=T(u(i))f$, that is, $T(A)$ is the infimum in $\underline{C}_0\cup\{C\}$, so that $\underline{C}_0\cup\{C\}=\underline{C}_0$ by (ii).

(iv) Consider any diagrams Θ: $K\to\underline{C}_0$ with $\inf \Theta=(C', t)$ in $\underline{C}$, where t is a natural equivalence and D: $I\to\underline{A}$ with $\inf D=(A, u)$. Then $T(A)\to TD(i)$ is a natural in $i\in I$. Let $x(i)$: $C'\to TD(i)$ be a natural in $i\in I$, for each k of K then we have a natural map $x(i)t^{-1}(k)$: $\Theta(k)\to TD(i)$ in i of I. There exists a unique map $y(k)$: $\Theta(k)\to T(A)$

such that $T(u(i))\cdot y(k)=x(i)t^{-1}(k)$. Hence $T(u(i))\cdot y(k)t(k)=x(i)$.

$$\begin{array}{ccc} \theta(k) & \xrightarrow{\;t^{-1}(k)\;} & C' \\ {\scriptstyle y(k)}\downarrow & & \downarrow{\scriptstyle x(i)} \\ T(A) & \xrightarrow[\;T(u(i))\;]{} & TD(i) \end{array} \quad \text{commutes.}$$

By (ii), C' must be in $\underline{C}_0$.

3. Categery of inf-preserving functors. Let $\underline{A}$ be any small category. We shall write $[\underline{A},\ Ens]_{\text{inf}}$ for the category of all inf-preserving functors from $\underline{A}$ to *Ens*.

LEMMA 3. *For each object C of $\underline{C}$ and each functor S: $\underline{C}\to$Ens, there is a bijection ψ: $S(c)\cong[h_c S]$, where h_c is a functor $[C,\quad]$: $\underline{C}\to$Ens and the map ψ is defined for $x\in S(c)$ and $f\in h_c(A)$ for A of $\underline{C}$ as $(\psi(x))(f)=(S(f))(x)$,* [2].

By the lemma 1 the cannonical embedding H: $\underline{A}\to[\underline{A},\ Ens]^{\circ}$ induce the embedding of $\underline{A}$ into $[\underline{A},\ Ens]_{\text{inf}}$. Using the proposition 2, we shall show the following theorem.

THEOREM. (1) *The category $[A,\ Ens]^{\circ}_{\text{inf}}$ is sup-complete.*

(2) *For any diagram θ with* $\inf\theta=(z,t)$ *in $[\underline{A},\ Ens]^{\circ}$ if t is a natural equivalence then $[\underline{A},\ Ens]_{\text{inf}}$ is inf-complete.*

Proof. (1) Let $\underline{B}$ be the category of all functors T: $\underline{A}\to Ens$ in $[\underline{A},\ Ens]^{\circ}$ such that $[T, H(\ )]$ preserves infimums. By the lemma 3, $T\cong[T, H(\ )]$. Therefore we have $[\underline{A},\ Ens]^{\circ}_{\text{inf}}=\underline{B}$ by the lemma 2. Hence the category $\underline{B}$ is sup-complete by (iii) of the proposition 2.

(2) Since $[\underline{A},\ Ens]^{\circ}_{\text{inf}}=\underline{B}$, by (iv) o proposition 2 it follows that $[\underline{A},\ Ens]_{\text{inf}}$ is inf-complete.

References

[1] J. Lambeck, *Completions of categories*, Springer Verlag, Berlin, 1966.

[2] S. MacLane, *Categorical algebra*, Bull. Amer. Math. Soc. 71 (1965), 40-106.

[3] B. Mitchell, *Theory of categories*, Academic Press, New York, 1965.

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