An Extension of the Notion of Higher Differential Coefficients

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1. Definitions. Throughout this paper z is assumed to take any value in an intersection of a range $\alpha \leqq \arg z \leqq \beta$ $(\alpha \leqq \beta)$ and a circular neighborhood of $z=0$. Also functions of z are assumed to be defined on such a region. In particular when $\alpha=\beta=0$, z takes some positive real value or 0.

When a function $f(z)$ of z has differential coefficients in ordinary sense, let us denote them by

$$f'(z),\ f''(z),\ \cdots,\ f^{(n)}(z),\ \cdots.$$

If a function $f(z)$ has the n th differential coefficients at $z=0$, then

$$f^{(n)}(0)=\lim_{z\to 0}\{f^{(n-1)}(z)-f^{(n-1)}(0)\}/z.$$

Hence

$$f^{(n-1)}(z)=f^{(n-1)}(0)+f^{(n)}(0)z+o(z).$$

Integrating along the line segment joining 0 and z, we have

$$f^{(n-2)}(z)=f^{(n-2)}(0)+f^{(n-1)}(0)z/1!+f^{(n)}(0)z^2/2!+o(z^2).$$

Continuing integration $n-2$ times we have

$$f(z)=f(0)+f'(0)z/1!+f''(0)z^2/2!+\cdots+f^{(n)}(0)z^n/n!+o(z^n).$$

From this we have the following extended notion of n th differential coefficients.

DEFINITION 1. When a function $f(z)$ is represented as

$$f(z)=A_0+A_1z+A_2z^2+\cdots+A_nz^n+o(z^n) \qquad (1),$$

where A_i's are constants, $n!A_n$ is denoted by $D^nf(0)$. $D^nf(0)$ may be called *the generalized differential coefficient of the* 1*st kind* of $f(z)$ at $z=0$.

It is obvious that $D^nf(0)$ is unique whenever it exists. And from what we have seen,

THEOREM 1. *The existence of* $f^{(n)}(0)$ *implies the existence of* $D^nf(0)$. *And*

$$D^nf(0)=f^{(n)}(0).$$

For the difference of n th order of $f(z)$

$$\Delta^n(f,a,z)=\sum_{r=0}^{n}{}_nC_r(-1)^{n-r}f(a+rz).$$

DEFINITION 2. When $\lim_{z\to 0}\Delta^n(f,0,z)/z^n$ exists, this limit is denoted by $d^nf(0)$. $d^nf(0)$ may be called *the generalized differential coefficient of the* 2*nd kind* of $f(z)$ at $z=0$.

Since $\Delta^n(z^m,a,z)=m(m-1)\cdots(m-n+1)a^{m-n}z^n+z^{n+1}g(a,z)$, where $g(a,z)$ is either 0 or a homogeneous polynomial of $m-n-1$ th degree in a and z, (1) implies $d^nf(0)=n!A_n=D^nf(0)$. Hence

THEOREM 2. *The existence of* $D^nf(0)$ *implies the existence of* $d^nf(0)$. *And*

$$d^nf(0)=D^nf(0).$$

Clearly the converses to Theorems 1 and 2 are true when $n=1$, that is, the existence of $Df(0)$ and $df(0)$ imply that of $f'(0)$ and $Df(0)$ respectively. However we will see in Example 1, (b) and (c), that those converses are not true when $n\geqq 2$.

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The significance of $D^n f(0)$ and $d^n f(0)$ is that the existence of lower derivatives is not necessary to define $D^n f(0)$ and $d^n f(0)$, whereas the existence of $f^{(n-1)}(z)$ throughout some neighborhood of $z=0$ is necessary to define $f^{(n)}(0)$.

2. Example. The following theorems will easily be proved.

THEOREM 3. $D^n\{f(0)+g(0)\}=D^n f(0)+D^n g(0)$, $D^n\{cf(0)\}=cD^n f(0)$, $d^n\{f(0)+g(0)\}=d^n f(0)+d^n g(0)$, $d^n\{cf(0)\}=cd^n f(0)$, *whcre c is a constant.*

THEOREM 4. *The existence of $D^n f(0)$ implies the existence of $D^m f(0)$ for positive integer m less than n.*

This property that the existence of a higher differential coefficient implies that of lower differential coefficients does not necessarily hold for $d^n f(0)$ as we shall see in the following example.

EXAMPLE 1. For 0 or positive real value of z, let $f(z)=z^n$ when z is a rational number, $f(z)=0$ when z is an irrational number, where n is a positive integer, then

(a) when $n \geqq 1$, $d^{n+1}f(0)=0$, eventhough $d^n f(0)$ does not exist.

(b) when $n \geqq 3$, $D^{n-1}f(0)=0$, eventhough $f^{(n-1)}(0)$ does not exist.

(c) when $n \geqq 1$, $d^{n+1}f(0)=0$, eventhough $D^{n+1}(0)$ does not exist.

Proof. (a) Let $f_1(z)=z^n$, $f_2(z)=0$. Since z, $2z$, $3z$, $\cdots$, nz, $(n+1)z$ are either simultaneously rational numbers or simultaneously irrational numbers,

when z is rational,

$$\Delta^{n+1}(f, 0, z)=\Delta^{n+1}(f_1, 0, z)=0,$$
$$\Delta^n(f, 0, z)=\Delta^n(f_1, 0, z)=n!z^n,$$

when z is irrational,

$$\Delta^{n+1}(f, 0, z)=\Delta^{n+1}(f_2, 0, z)=0,$$
$$\Delta^n(f, 0, z)=\Delta^n(f_2, 0, z)=0.$$

Therefore

$$d^{n+1}f(0)=\lim_{z\to 0} \Delta^{n+1}(f, 0, z)/z^{n+1}=0,$$

eventhough

$$d^n f(0)=\lim_{z\to 0} \Delta^n(f, 0, z)/z^n$$

does not exist.

(b) Since $f(z)=o(z^{n-1})$, $D^{n-1}f(0)=0$ by Definition 1. On the other hand since $f(z)$ is discontinuous at every point except $z=0$, $f'(z)$ does not exist except $z=0$. Therefore $f''(0)$ does not exist, and furthermore $f^{(n-1)}(0)$ does not exist for $n \geqq 3$.

(c) $d^{n+1}f(0)=0$ by (a). Since $d^n f(0)$ does not exist by (a), $D^n f(0)$ does not exist by Theorem 2, and hence $D^{n+1}f(0)$ does not exist by Theorem 4. q.e.d.

3. Relations between D^n and d^n. As we saw in Example 1, (c), $D^n f(0)$ may not exist eventhough $d^n f(0)$ exists for $n \geqq 2$. Then there naturally arises a question, "Does $D^n f(0)$ exist when both $d^n f(0)$ and $D^{n-1}f(0)$ exist for $n \geqq 2$?" The answer is affirmative in the case $n=2$ (Theorem 5), whereas it is negative in the case $n=3$ (Example 2). It is an open question whether the answer is always negative for $n \geqq 3$ or not.

THEOREM 5. *$D^2 f(0)$ exists when both $d^2 f(0)$ and $Df(0)$ exist.*

Proof. If we write

$$f(z)=f(0)+\{df(0)\}z+\{d^2f(0)\}z^2/2+g(z),$$

then $$g(0)=dg(0)=d^2g(0)=0.$$

Hence $$g(z)=o(z)$$

and $$g(2z)-2g(z)=o(z^2) \quad (2).$$

Since $$g(2z)/2z-g(z)/z=o(z)$$

by (2), for an arbitrarily given $\varepsilon>0$, we can choose a positive number δ, so that when $|z|<\delta$ (3),

$$|g(2z)/2z-g(z)/z|<\varepsilon\,|z|.$$

For z of (3)

$$\left|g(z)/z-g\left(\frac{z}{2}\right)\ \frac{z}{2}\right|<\varepsilon\left|\frac{z}{2}\right|$$

$$\left|g\left(\frac{z}{2}\right)/\frac{z}{2}-g\left(\frac{z}{2^2}\right)/\frac{z}{2^2}\right|<\varepsilon\left|\frac{z}{2^2}\right|$$

....

$$\left|g\left(\frac{z}{2^{k-1}}\right)/\frac{z}{2^{k-1}}-g\left(\frac{z}{2^k}\right)/\frac{z}{2^k}\right|<\varepsilon\left|\frac{z}{2^k}\right|.$$

Hence

$$\left|g(z)/z-g\left(\frac{z}{2^k}\right)/\frac{z}{2^k}\right|<\varepsilon\,|z|.$$

Since $\lim_{k\to\infty} g\left(\frac{z}{2^k}\right)/\frac{z}{2^k}=0$ by $g(z)=o(z)$, we have

$$|g(z)/z|<\varepsilon|z|,$$

that is $$g(z)=o(z^2).$$ q. e. d.

EXAMPLE 2. Let $g(z)=2^p\left(\frac{7}{3}\right)^q$ when $z=2^p3^q$ $(p,q=0,\pm1,\pm2,\cdots)$, and $g(z)=0$ for any other positive real value of z and $z=0$. If $f(z)=z^2g(z)$, then $d^3f(0)=0$, $D^2f(0)=0$, eventhough $D^3f(0)$ does not exist.

Proof. Since $g(z)\to0$ when $z\to0$, $g(z)=0(1)$.

Hence $$f(z)=o(z^2) \quad (4)$$

and $$D^2f(0)=0.$$

When

$$z=2^p3^q,\quad 3g(3z)-4g(2z)+g(z)=3g(2^p3^{q+1})-4g(2^{p+1}3^q)+g(2^p3^q)=0.$$

Therefore, for any value of z

$$3g(3z)-4g(2z)+g(z)=0.$$

From this we have

$$f(3z)-3f(2z)+3f(z)=0.$$

That is $$\Delta^3(f,0,z)=0,$$

therefore $$d^3f(0)=0.$$

However when $z=2^p$ $\quad f(z)/z^3=g(z)/z=2^p/2^p=1,$

and we have $$f(z)\neq o(z^3) \quad (5).$$

If $D^3f(0)$ were to exist, $D^3f(0)$ would be equal to $d^3f(0)=0$. From this and (4), we would have $f(z)=o(z^3)$ contrary to (5). So $D^3f(0)$ does not exist. q.e.d.

4. Leibniz' formula. It is easy to see that the following formula holds.

THEOREM 6. $D^n\{f(0)g(0)\}=\sum_{r=0}^{n}{}_nC_r\, D^rf(0)D^{n-r}g(0)$, *when differential coefficients of the right hand side exist.*

It is an open question whether Leibniz' formula holds for d^n or not. However we can prove the following theorem.

THEOREM 7. *For* $g(z)=z^m$ $(m=1, 2, 3,\cdots)$,

$$d^n\{f(0)g(0)\}=\sum_{r=0}^{n}{}_nC_r\, d^rf(0)d^{n-r}g(0) \qquad (6),$$

when $df(0)$, $d^2f(0)$, $\cdots$, $d^nf(0)$ *exist* $(n\geqq 1)$.

In order to prove this theorem, we shall first prove the following lemmas.

LEMMA 1. *When* $d^{n-1}f(0)$, $d^nf(0)$ *exist*, $d^nh(0)=nd^{n-1}f(0)$ $(n\geqq 1)$, *where*

$$h(z)=zf(z).$$

Proof. From the assumptions

$$\Delta^{n-1}(f,0,z)=\sum_{r=0}^{n-1}{}_{n-1}C_r(-1)^{n-1-r}f(rz)=o(z^{n-1}),$$

$$\Delta^n(f,0,z)=\sum_{r=0}^{n}{}_nC_r(-1)^{n-r}f(rz)=\{d^{n-1}f(0)\}z^{n-1}+o(z^{n-1}).$$

Multiplying the sum of the above two exprssions by n, we have

$$\sum_{r=0}^{n} r_nC_r(-1)^{n-r}f(rz)=\{nd^{n-1}f(0)\}z^{n-1}+o(z^{n-1}).$$

Hence

$$\Delta^n(h,0,z)=z\sum_{r=0}^{n} r_nC_r(-1)^{n-r}f(rz)=\{nd^{n-1}f(0)\}z^n+o(z^n),$$

therefore $\quad d^nh(0)=\lim_{z\to 0}\Delta^n(h,0,z)/z^n=nd^{n-1}f(0).$ q. e. d.

LEMMA 2. *When* $f(0)=df(0)=d^2f(0)=\cdots\cdots=d^nf(0)=0$,

$$h(0)=dh(0)=d^2h(0)=\cdots\cdots=d^nh(0)=0 \ (n\geqq 1),$$

where $h(z)=z^mf(z)$ $(m=1,2,\cdots)$.

Proof. Proof is complete by induction if we prove the lemma for $m=1$.

When $h(z)=zf(z)$, $h(0)=0$ is obvious, and if we replace n of Lemma 1 by $1, 2, 3, \cdots, n$ successively, we get $dh(0)=0$, $d^2h(0)=0$, $d^3h(0)=0$, $\cdots$, $d^nh(0)=0$.

Proof of Theorem 7. If we write

$$f(z)=f(0)+\{df(0)\}z/1!+\{d^2f(0)\}z^2/2!+\cdots+\{d^nf(0)\}z^n/n!+F(z),$$

then

$$F(0)=dF(0)=d^2F(0)=\cdots=d^nF(0)=0,$$

and

$$f(z)g(z)=f(0)z^m+\{df(0)\}z^{m+1}/1!+\{d^2f(0)\}z^{m+2}/2!+\cdots,$$
$$+\{d^nf(0)\}z^{m+n}/n!+z^mF(z).$$

Since for the last term $h(z)=z^mF(z)$ we have $d^nh(0)=0$ by Lemma 2,

$$d^n\{f(0)g(0)\}=0 \text{ when } m>n,$$

$$d^n\{f(0)g(0)\}=n!d^{n-m}f(0)/(n-m)! \text{ when } m\leqq n,$$

which are nothing but (6). q.e.d.

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