# ON DIMENSION OF NON-METRIZABLE SPACES

CHINHONG PARK

**1. Introduction.** This paper is a close connection of "some remarks on dimension of topological spaces". The purpose of this note is to extend some results of the theory of metric spaces to somewhat more general spaces. Except for some facts we shall give results in the case of non-metrizable spaces.

Theorem 1 and 2 describe the dimension theory in non-metrizable spaces. We shall prove these by equivalence. Proposition 2 and 3 are a theory of normal families to establish another dimension theory on the new concept of dimension. This is to give a short sketch of normal family to re-establish the dimension theory. Here we shall define a closed-normal family. By this definition, each family of all hereditarily paracompact spaces and paracompact spaces with Ind $\leq$ n, dim $\leq$ n respectively is a closednormal family.

**2. closed-normal families and dimension of non-metizable spaces.**

Before we consider the issues in non-metrizable spaces, To begine with, let us show that the following proposition is held.

PROPOSITION 1. *In a metric space R let C be a closed set of a space R. For every continuous mapping* $f_i$ *of C into* $S^n$ *if R-C has covering dimension* $\leq n$ *and C has covering dimension* $\leq$ n-1, *there exist continuous extensions of* $f_i$ *and* $f_j$ *over R which are homotopic each other. Where* $f_j$ *is arbitrary continuous mapping of C into* $S^n$.

Next, we are to give the brief theory of normal families to establish another dimension theory on the new concept of dimension. The theory of normal families was first established by W.Hurewicz to deduce systematically fundamental theorems of dimension theory for separable metric spaces and was extended by K.Morita to non-separable metric spaces. This is to re-establish a short result of dimension theory by use of normal families.

PROPOSITION 2. *Let us define that a normal family C contains a metric space R if and if R has strong inductive dimension* $\leq n$. *Then all the metric spaces with* $Ind \leq n$ *make a normal family.*

Proof. Let C be a family of all the metric spaces with Ind $\leq$ n. To show C is a normal family, Let

$$S \subset R \subset C$$

Then by definition of hypothesis

$$\text{Ind } R \leqq n.$$

Hence by virtue of Theorem II. 7 of [1]

$$\text{Ind } S \leqq n$$

This means S $\in$C.

Let $\{F_\gamma | \gamma \in \Gamma\}$ be a locally finite closed covering of R such that

$$\text{Ind } F_\gamma \in C \text{ for every } \gamma \in \Gamma$$

then Ind $F_\gamma \in C$ means Ind $F_\gamma \leqq n$.

Hence $R \in C$.

Similarly if $\{F_i | i=1, 2, \cdots\cdots\}$ is a closed covering of a metric space R such that

$$F_i \in C, \quad i=1, 2, \cdots\cdots,$$

then $R \in C$.

We shall give another definition in order to construct the concept of a normal family in more general spaces as well as metric spaces.

DEFINITION. A family C-n of topological spaces is called a closed-normal family if it satisfies.

i) if for every closed subset S of a topological space R, $S \subset R$ and $R \in$ C-n, then $S \in$ C-n.

ii) if $\{F_\gamma | \gamma \in \Gamma\}$ is a locally countable closed covering of R such that

$$F_\gamma \in \text{C-n for every } \gamma \in \Gamma,$$

then $R \in$ C-n.

By virtue of the definition above, a family $\mathfrak{F}$ of metric spaces is a closed-normal family if and only if it satisfies

(1) if for every closed subset S of a metric space R, $S \subset R$ and $R \in \mathfrak{F}$, then $S \in \mathfrak{F}$.

(2) if $\{F_i | i=1, 2, \cdots\cdots\}$ is a closed covering of R such that

$$F_i \in \mathfrak{F}, \quad i=1, 2, \cdots\cdots$$

then $R \in \mathfrak{F}$.

(3) if $\{F_\gamma | \gamma \in \Gamma\}$ is a locally finite closed covering of R such that

$$F_\gamma \in \mathfrak{F} \text{ for every } \gamma \in \Gamma,$$

then $R \in \mathfrak{F}$.

Now we are ready to apply the concept of closed-normal family to dimension theory in hereditarily paracompact and normal spaces.

PROPOSITION 3. *All hereditarily paracomact spaces and paracompact spaces with* Ind $\leq n$, dim $\leq n$ *respectively make a closed-normal family.*

Proof. Let $\mathfrak{F}$ be a family of all hereditarily paracompact spaces with Ind $R \leqq n$. Generally in a topological space R, if F is a closed subset of R, then

$$\text{Ind } F \leq \text{Ind } R$$

By use of this fact, for every closed subset S of a hereditarily paracompact space R,

If

$$S \subset R \in \mathfrak{F},$$

since

$$\text{Ind } R \leq n,$$

therefore

$$\text{Ind } S \leq n.$$

This means $S \in \mathfrak{F}$.

Let $\{F_\gamma | \gamma \in \Gamma\}$ be a locally countable closed covering of R such that

$$F_\gamma \in \mathfrak{F} \text{ for every } \gamma \in \Gamma.$$

In view of proposition 4 of [2] and Ind $F_\gamma \leq n$, $R \in \mathfrak{F}$. Hence $\mathfrak{F}$ is a closed-normal family.

Next, let $\mathfrak{C}$ be a family of all paracompact spaces with dim $R \leqq n$.

Suppose

$$S \subset R \in \mathfrak{C}.$$

for every closed subset S of a paracompact space R. By virtue of proposition VII.2 of (1) and dim S $\leqq$ n, it is obvious that

$$S \in \mathfrak{C}.$$

If $\{G_\gamma | \gamma \in \Gamma\}$ is a locally countable closed covering of a paracompact space R such that

$$G_\gamma \in \mathfrak{C} \text{ for every } \gamma \in \Gamma,$$

then dim R $\leqq$ n because dim $G_\gamma \leqq$ n and proposition 4 of (2) This implies

$$R \in \mathfrak{C}.$$

Also in Proposition 2, all the metric spaces with Ind $\leqq$ n make a closed-normal family. The following Lemma is satisfied in a general topological sapce, we will use this Lemma in the process of proof.

LEMMA 1. Let $\mathfrak{F} = \{V_\alpha | \alpha < \tau\}$ *be an open collection of a topological space R such that*

$$\text{ord } B(\mathfrak{F}) \leqq n$$

Let

$$F_0 = \overline{V}_0$$

$$F_\alpha = \overline{V}_\alpha - \bigcup \{V_\beta | \beta < \alpha\} \text{ for every } \alpha < \tau.$$

*Then* $F = \{F_\alpha | \alpha < \tau\}$ *is a closed collection* with

$$\text{ord } F \leqq n+1.$$

Now we are ready to consider dimension theory in non-metrizable spaces. It is difficult to establish a dimension theory for normal space or even for paracompact $T_2$-spaces. The following statements are brief results obtained in relations between dimension theory of metric spaces and of non-metrizable spaces. Throughout the following facts we consider only $T_2$-spaces which satisfy Hausdorff's separation axiom.

THEOREM 1. *Let R be a normal space. The following properties are equivalent.*

(1) dim R $\leqq$ n

(2) *For every locally finite open collection* $\{U_\gamma | \gamma \in \Gamma\}$ *in R and closed collection* $\{F_\gamma | \gamma \in \Gamma\}$ *in R such that* $F_\gamma \subset U_\gamma$ *for every* $\gamma$ *there exists an open collection* $\{V_\gamma | \gamma \in \Gamma\}$ *in R such that*

$$F_\gamma \subset V_\gamma \subset \overline{V}_\gamma \subset U_\gamma$$

$$\text{ord } \{B(V_\gamma) | \gamma \in \Gamma\} \leqq n.$$

(3) *For every open covering* $\{U_i | i=1, 2, \cdots\cdots, k\}$ *there exists a closed covering* $\mathfrak{F} = \{F_i | i=1, 2, \cdots, k\}$ *of R such that*

$$F_i \subset U_i, \ i=1, 2, \cdots\cdots, k$$

$$\text{ord } \mathfrak{F} \leqq n+1$$

(4) *For every open covering* $\{U_i | i=1, 2, \cdots\cdots, k\}$ *there exists an open covering* $\mathfrak{L} = \{V_i | i=1, 2, \cdots\cdots, k\}$ *such that*

$$V_i \subset U_i, \ i=1, 2, \cdots\cdots, k$$

$$\text{ord } \mathfrak{L} \leqq n+1.$$

Proof. (1) implies (2); Let $\mathfrak{U}_\gamma$ be the binary covering $\{U_\gamma, R\text{-}F_\gamma\}$. Then, since $\{U_\gamma | \gamma \in \Gamma\}$ is locally finite, $\bigwedge \{\mathfrak{U}_\gamma | \gamma \in \Gamma\}$ is a locally finite open covering, By virtue of the first part of the poof in proposition II. 5. B of (1), we can easily see the statement holds.

(2) implies (3); For every open covering $\{U_i | i=1, 2, \cdots\cdots, k\}$ since R is normal, there exists a closed covering $\{G_i | i=1, 2, \cdots\cdots, k\}$ such that

$$G_i \subset U_i, \ i=1, 2, \cdots\cdots, k.$$

By use of hypothesis there exists an open covering $\{V_i | i=1,2,\cdots\cdots,k\}$ such that

$$G_i \subset V_i \subset \overline{V}_i \subset U_i$$
$$\text{ord } \{B(\overline{V}_i) | i=1,2,\cdots\cdots,k\} \leqq n$$

Let

$$F_i = \overline{V}_i - \bigcup_{j=1}^{i=1} V_j$$

By use of Lemma 1 $\mathfrak{F} = \{F_i | i=1,2,\cdots\cdots,k\}$ is the desired set.

(3) implies (4) and (4) implies (1): we can easily see that statements hold.

Next we are ready to obtain a similar result in paracompact spaces as a dimension theory of metric spaces. By argument of equivalence we will prove the following statements. In metric spaces this was proved.

THEOREM 2. *In a paracompact space R the following properties are equivalent.*

(1) $\dim R \leq n$

(2) *For every locally finite open collection* $\{U_\gamma | \gamma < \tau\}$ *in R and closed collection* $\{F_\gamma | \gamma < \tau\}$ *such that*

$$F_\gamma \subset U_\gamma \text{ for every } \gamma < \tau$$

*there exists an open collection* $\{V_\gamma | \gamma < \tau\}$ *in R such that*

$$F_\gamma \subset V_\gamma \subset \overline{V}_\gamma \subset U_\gamma$$
$$\text{ord } \{B(V_\gamma) | \gamma < \tau\} \leq n.$$

(3) *For every locally finite open covering* $\{U_\gamma | \gamma < \tau\}$ *there exists a closed covering* $\mathfrak{F} = \{F_\gamma | \gamma < \tau\}$ *such that*

$$F_\gamma \subset U_\gamma \text{ for every } \gamma < \tau$$
$$\text{ord}\,\mathfrak{F} \leq n+1.$$

(4) *For every locally finite open covering* $\{U_\gamma | \gamma < \tau\}$ *of R there exists an open covering* $\mathfrak{V} = \{V_\gamma | \gamma < \tau\}$ *such that*

$$V_\gamma \subset U_\gamma$$
$$\text{ord } \mathfrak{V} \leq n+1.$$

Proof. (1) implies (2): By virtue of Theorem 1, we can easily see that the statement is held. (2) implies (3): For every locally finite open covering $\{U_\gamma | \gamma < \tau\}$ since R is normal, there exists a closed covering $\{G_\gamma | \gamma < \tau\}$ such that

$$G_\gamma \subset U_\gamma \text{ for every } \gamma < \tau.$$

By use of hypothesis there exists an open covering $\{V_\gamma | \gamma < \tau\}$ such that

$$G_\gamma \subset V_\gamma \subset \overline{V}_\gamma \subset U_\gamma$$
$$\text{ord } \{B(V_\gamma) | \gamma < \tau\} \leq n.$$

Let

$$F_0 = \overline{V}_0$$
$$F_\gamma = \overline{V}_\gamma - \bigcup \{V_\alpha | \alpha < \gamma\}$$
$$\text{for every } \gamma < \tau.$$

Then by virtue of Lemma 1, $\mathfrak{F} = \{F_\gamma | \gamma < \tau\}$ is the desired set.

(3) implies (4): In a view of hypothesis, for every locally finite open covering $\{U_\gamma | \gamma < \tau\}$ of R we can obtain a locally finite closed covering $\mathfrak{F} = \{F_\gamma | \gamma < \tau\}$ such that

$$F_\gamma \subset U_\gamma \text{ for every } \gamma < \tau$$
$$\text{ord } \mathfrak{F} \leq n+1.$$

then there exists an open covering $\mathfrak{C}$ such that each elements of $\mathfrak{C}$ intersets at most $n+1$ elements of $\mathfrak{F}$. Since $\mathbb{R}$ is fully normal, we can choose an open covering $\mathfrak{D}$ such that

$$\mathfrak{D}^* < \mathfrak{C}$$

Let

$$\mathfrak{L} = \{V_\gamma | \gamma < \tau\},$$
$$V_\gamma = S(F_\gamma, \ \mathfrak{D}) \cap U_\gamma$$

then $\mathfrak{L} = \{W_\gamma | \gamma < \tau\}$ is the desired open covering.

We shall show that

$$\mathfrak{L} = \{V_\gamma | \gamma < \tau\}$$

is an open covering of order $\leq n+1$. Suppose

$$\bigcap_{i=1}^{n+2} V_{\gamma_i} \neq \phi.$$

where

$$V_{\gamma_i} = S(F_{\gamma_i}, \mathfrak{D}) \cap U_{\gamma_i}.$$

Let

$$x \in V_{\gamma_i} = S(F_{\gamma_i}, \mathfrak{D}) \cap U_{\gamma_i},$$
$$i = 1, 2, \cdots\cdots, n+2.$$

then

$$x \in S(F_{\gamma_i}, \ \mathfrak{D}) \text{ and } x \in U_{\gamma_i},$$

since

$$S(F_{\gamma_i}, \mathfrak{D}) = \cup \{D \in \mathfrak{D} \,|\, F_{\gamma_i} \cap D \neq \phi.\} \quad i = 1, 2, \cdots\cdots, n+2$$

there exists a member $D_j$ of $\mathfrak{D}$ such that

$$F_{\gamma_i} \cap D_j \neq \phi \text{ and } x \in D_j,$$
$$i = 1, 2, \cdots\cdots, n+2$$

since

$$(\bigcup_{j=1}^{n+2} D_j) \cap F_{\gamma_i} \neq \phi$$

$$\text{and } x \in \bigcup_{j=1}^{n+2} D_j$$

We have

$$\bigcup_{j=1}^{n+2} D_j \in \mathfrak{D}^\Delta < \mathfrak{D}^* < \mathfrak{C}.$$

Hence there exists a member $S(D, \mathfrak{D})$ of $\mathfrak{D}^*$ such that

$$\bigcup_{j=1}^{n+2} D_j \subset S(D, \ \mathfrak{D})$$

By virtue of

$$\mathfrak{D}^* < \mathfrak{C}.$$

We have a member $C$ of $\mathfrak{C}$ satisfying

$$S(D, \mathfrak{D}) \subset C$$

therefore

$$\bigcup_{j=1}^{n+2} D_j \subset S(D, \mathfrak{D}) \subset C$$

and $C \cap F_{\gamma_i} \neq \phi$, $i=1, 2, \cdots\cdots, n+2$

This is a contradiction.

(4) implies (1): By virtue 2.1 it is clear.

COROLARY. *Let R be a paracompact. For every open covering* $\{U_\gamma | \gamma \in \Gamma\}$ *of R there exists a closed covering* $\mathfrak{F} = \{F_\gamma | \gamma \in \Gamma\}$ *such that*

$$F_\gamma \subset U_\gamma$$

$$\text{ord } \mathfrak{F} \leq n+1.$$

Then

$$\dim R \leq n.$$

Proof. For any locally finite open covering $\{U_\gamma | \gamma \in \Gamma\}$ of R, by use of hypothesis there exists a closed covering $\mathfrak{F} = \{F_\gamma | \gamma \in \Gamma\}$ such that

$$F_\gamma \subset U_\gamma$$

$$\text{ord } \mathfrak{F} \leq n+1.$$

Therefore

$$\dim R \leq n.$$

## REFERENCES

1. J.Nagata, Modern Dimension Theory, Interscience, New York, 1965.
2. C.H.Park, Some Remarks on Dimensions of Topological spaces. K.M.S. Journal. 1
3. J.L.Kelly, General topology, Van Nostrand, 1955.