A NOTE ON HYPERSURFACES OF ALMOST CONTACT MANIFOLDS

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1. Introduction.

In a recent paper [1], the authors consider a $2n$-dimensional manifold M imbedded in almost contact manifold $\tilde{M}^{2n+1}$ with fundamental affine collineation ϕ, fundamental vector field ξ and contact form η, and assume that for each $p \in M$ the vector field ξ does not belong to the tangent hyperplane of the hypersurface. This means that the vector field ξ can be taken as the "affine normal" to the hypersurface.

More recently [2], in the case which ξ is always tangent to M, it is known that there exists a vector field N playing the role of "affine normal" along the hypersurface.

In this paper, we consider the case where ξ is always tangent to M.

2. Hypersurfaces of almost contact manifolds.

Let $\tilde{M}=\tilde{M}^{2n+1}(\phi,\xi,\eta)$ be an almost contact manifold, and let $M=M^{2n}$ be a hypersurface imbedded in $\tilde{M}$. Throughout this paper, we assume that the vector field ξ is always tangent to M. Then it is known that a vector field N exists along the hypersurface M such that

$$\phi N=-A,\ \ \eta(N)=0 \quad (2.1)$$
$$\text{and } \phi X=fX+\alpha(X)\cdot N$$

for some vector field A on M, (1, 1) type tensor field f and 1-form α.

Applying ϕ to the relation (2.1), we get

$$-X+\eta(X)\xi=f^2X+\alpha(fX)N-\alpha(X)A,$$

which shows that

$$f^2=-I+\eta\otimes\xi+\alpha\otimes A,$$
$$\alpha(fX)=0,\ \ \eta(A)=0,\ \ \alpha(\xi)=0,\ \ \alpha(A)=1. \quad (2.2)$$
$$f(\xi)=0,\ \ f(A)=0,\ \ \eta(fX)=0.$$

for any $X\in\mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the set of all vector fields on M. Thus we

Manifolds, mapping, tensor fields and any geometric objects we discuss are assumed to be differentiable and of class C^∞.

have that, in an almost contact manifold $\tilde{M}$, a hypersurface M for the vector field ξ to be tangent to M admits $(f,\xi,A,\eta,\alpha,\lambda)$—structure.

Moreover, if we define a tensor field $\tilde{f}$ as

(2.3) $$\tilde{f}=f+\eta\otimes A,$$

then we obtain

$$\tilde{f}^2(X)=f^2(X)+\eta(fX)A+\eta(X)f(A)+\eta(X)\eta(A)A=f^2(X),$$

that is, $\tilde{f}^2=f^2$ on M. From which we have that

$$\tilde{f}^4(X)=-\tilde{f}^2(X) \qquad \text{on } M,$$

by virtue of (2.2). Since $\tilde{f}$ has the same rank at each point of M, we find that the tensor field $\tilde{f}$ defined as (2.3) is a quartic structure in M.

On the other hand, for the same $\tilde{f}$ we get

$$\tilde{f}^2=-I+\eta\otimes\xi+\alpha\otimes A,$$

and $\eta(A)=0,\ \alpha(\xi)=0.$

Hence we can see that the hypersurface M is to be globally framed.

Combining the above results.

THEOREM 1. *The hypersurface M imbedded in almost contact manifold $\tilde{M}$ in such a way that the vector field ξ is always tangent to M is a globally framed quartic manifold.*

3. Hypersurfaces of Sasakian manifolds.

Let $\tilde{M}=\tilde{M}^{2n+1}(\phi,\xi,\eta,\tilde{g})$ be an almost contact manifold, and let $\tilde{\nabla}$ be the Riemannian connection of $\tilde{g}$. For $X, X\in\mathfrak{X}(M)$, we get

(3.1) $$\tilde{\nabla}_XY=\nabla_XY+h(X,Y)N,$$

(3.2) $$\tilde{\nabla}_XN=-HX+w(X)N,$$

where ∇_XY and $-HX$ are the tangential parts (with respect to N) of $\tilde{\nabla}_XY$ and $\tilde{\nabla}_XN$, respectively, to M. We can see that $\nabla:(X,Y)\to\nabla_XY$ is a symmetric connection on M, h is symmetric, and is called the second fundamental form of M (with respect to N).

If $h=0$ on M, then M is called to be totally geodesic. Let g be the induced metric: $g=\tilde{g}/M$. In general, the connection ∇ is not the Levi-Civita connection of g. Using (3.1) and (3.2), we obtain

(3.3) $$(\nabla_Xg)(Y,Z)=h(X,Y)g(N,Z)+h(X,Z)g(Y,N)$$

Suppose that ∇ is the Levi-Civita connection of the induced metric g, then we find

$$2g(\nabla_XY,Z)=X\cdot g(Y,Z)+Y\cdot g(X,Z)-Z\cdot g(X,Y)+g([X,Y],Z)$$

$$+g([Z,X],Y)+g(X,[Z,Y])$$
$$=2\tilde{g}(\nabla_X Y+h(X,Y)N,Z).$$

From which we have

(3.4) $$h(X,Y)\tilde{g}(N,Z)=0,$$

for any $X,Y,Z\in\mathfrak{X}(M)$. Since N is an affine normal, we find from (3.3) and (3.4) the following:

THEOREM 2. *In order that the connection ∇ of the hypersurface M imbedded in almost contact manifold $\tilde{M}$ in such a way that the vector field ξ is always tangent to M is a Riemannian connection of $g=\tilde{g}/M$, it is necessary and sufficient that M is totally geodesic.*

Now, we assume that $\tilde{M}=\tilde{M}^{2n+1}(\phi,\xi,\eta,\tilde{g})$ is a Sasakian manifold; that is, the following holds good:

(3.5) $$(\tilde{\nabla}_U\phi)V=\eta(V)U-\tilde{g}(U,V)\xi,\quad U,V\in\mathfrak{X}(\tilde{M}),$$

where $\mathfrak{X}(\tilde{M})$ is the set of all vector fields on $\tilde{M}$. It is well known that (3.5) implies

(3.6) $$\tilde{\nabla}_U\xi=\phi U$$

Suppose M is totally geodesic, then (3.6) implies $\alpha(X)=0$ for any X. Thus we have immediately the following:

THEOREM 3. *There are no totally geodesic hypersurfaces imbedded in a Sasakian manifold $\tilde{M}$ in such a way that the vector field ξ is always tangent to M.*

Summing up theorem 2 and 3, we obtain

THEOREM 4. *The induced connection ∇ of the hypersurface M imbedded in a Sasakian manifold $\tilde{M}$ in such a way that the vector field ξ is always tangent to M cannot be a Riemannian connection of $g=\tilde{g}/M$.*

4. Hypersurfaces of affinely cosymplectic manifold.

We assume that $\tilde{M}=\tilde{M}^{2n+1}(\phi,\xi,\eta)$ is affinely cosymplectic; an almost contact manifold $\tilde{M}^{2n+1}(\phi,\xi,\eta)$ with a symmetric affine connection $\tilde{\nabla}$ satisfies

$$\tilde{\nabla}\phi=0,\quad \tilde{\nabla}\eta=0.$$

Then we have

(4.1) $$(\nabla_X\eta)Y=0,\quad X,Y\in\mathfrak{X}(M),$$

(4.2) $$\tilde{\nabla}_X\phi Y=f(\nabla_X Y)+\alpha(\nabla_X Y)N-h(X,Y)A,\quad X,Y\in\mathfrak{X}(M).$$

On the other hand

(4, 3) $$\tilde{\nabla}_X \phi Y = (\nabla_X f)Y + f(\nabla_X Y) + h(X, fY)N - \alpha(Y)HX + \alpha(Y)\omega(X)N + (\nabla_X \alpha Y)N.$$

Comparing (4.2) and (4.3), we get

(4.4) $$(\nabla_X f)Y = \alpha(Y)HX - h(X, Y)A,$$

(4.5) $$(\nabla_X \alpha)Y = -h(X, fY) - \alpha(Y)\omega(X).$$

Moreover, we find from (4,1), (4.4) and (4.5)

(4.6) $$[f, f](X, Y) + d\eta(X, Y)\xi + d\alpha(X, Y)A = \alpha(Y)HfX - \alpha(X)HfY + \alpha(X)fHY - \alpha(Y)fHX + (\alpha\wedge\omega)(X, Y)A$$

Now we assume that M is to be totally flat, then $HX=0$.

Thus we find from (4.6) the following:

THEOREM 5. *Suppose that the hypersurface M imbedded in an affinely cosymplectic manifold $\tilde{M}$ in such a way that the vector field ξ is always tangent to M is to be totally flat. Then the necessary and sufficient condition in order that $(f, \xi, A, \eta, \alpha, \lambda)$—structure is normal is $\alpha\wedge\omega=0$ on M.*

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REFERENCE

[1] Samuel I. Goldberg and Kentaro Yano, *Noninvariant hypersurfaces of almost contact manifolds*. Kodai Math. Sem. Rep. 22(1970), 199—218.

[2] ——, *Polynomial structure on manifolds*. J. Math. Soc. Japan 22(1970), 25—34.