OPERATIONAL REPRESENTATIONS IN UNILATERAL AND BILATERAL OPERATIONAL CALCULUS

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1. The integral equation

$$\phi(p)=p\int_0^\infty e^{-pt}f(t)dt, \qquad R(p)>0 \tag{1.1}$$

represents the classical Laplace transform and the function $\phi(p)$ and $f(t)$ related by (1.1), are said to be operationally related to each other. $\phi(p)$ is called the image and $f(t)$ the original. Symbolically we can write

$$\phi(p)\doteqdot f(t) \text{ or } f(t)\doteqdot\phi(p)$$

and the symbol $\doteqdot$ is called *'operational'*.

Similarly we shall denote in two variables as:

$$\phi(p, q) \doteqdot f(x, y).$$

In the present paper, we shall obtain certain new correspondences between the original and the image in one and two variables. The methods are applied of Mellin transform and of operational calculus. The notations are applied of Ditkin and Prudnikov's operational calculus. we have also derived some infinite series.

2. (a). If $\phi(p)\doteqdot f(t)$; then

$$\phi(\sqrt{p})\doteqdot\frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-x^2/4t}f(x)dx \tag{2.1}$$

(i) consider $\phi(p)=\frac{\Gamma(v)}{2}p\left[(p-\sqrt{a})^{-v}\pm(p+\sqrt{a})^{-v}\right]$

$$\doteqdot t^{v-1}\begin{matrix}\sinh\\ \cosh\end{matrix}(\sqrt{at})\equiv f(t)$$

Hence from (2.1), we get

$$\sqrt{p}\left[\frac{1}{(\sqrt{p}-\sqrt{a})^v}\mp\frac{1}{(\sqrt{p}+\sqrt{a})^v}\right]\doteqdot\frac{2}{\Gamma(v)\sqrt{\pi t}}\int_0^\infty x^{v-1}e^{-x^2/4t}\begin{matrix}\sinh\\ \cosh\end{matrix}(\sqrt{a}x)dx$$

or, $$\sqrt{p}\left[(\sqrt{p}-\sqrt{a})^{-v}\mp(\sqrt{p}+\sqrt{a})^{-v}\right]\doteqdot\frac{2^{v/2}t^{(v-1)/2}}{\sqrt{\pi}}e^{\frac{at}{2}}\cdot\left[D_{-v}(-\sqrt{2at})\mp D_{-v}(\sqrt{2at})\right], \tag{2.2}$$

$$R(v)>0.$$

Also $$\frac{\sqrt{p}}{(\sqrt{p}+\sqrt{a})^v} \doteqdot \frac{2^{\frac{v}{2}} t^{\frac{v-1}{2}}}{\sqrt{\pi}} e^{\frac{at}{2}} D_{-v}(\sqrt{2at}),$$

or, $$\frac{\sqrt{p}\, a^{\frac{v}{2}}}{(\sqrt{p}+\sqrt{a})^{v+1}} \doteqdot \sqrt{\frac{2}{\pi}} (2t)^{\frac{v}{2}} e^{\frac{at}{2}} D_{-v-1}(\sqrt{2at}).$$

or, $$\frac{\sqrt{p}}{\sqrt{p}+\sqrt{a}} \cdot \frac{1}{1-\dfrac{\sqrt{a}}{\sqrt{p}+\sqrt{a}}} \doteqdot \sqrt{\frac{2}{\pi}} \sum_{v=0}^{\infty} (2t)^{\frac{v}{2}} a^{\frac{v}{2}} e^{\frac{at}{2}} D_{-v-1}(\sqrt{2at}).$$

Therefore $$\sum_{v=0}^{\infty} D_{-v-1}(\sqrt{2at})(2at)^{\frac{v}{2}} = \sqrt{\frac{\pi}{2}} e^{-\frac{1}{2}at} \tag{2.3}$$

Also $$\frac{\sqrt{p}}{\sqrt{p}+2\sqrt{a}} \doteqdot \sqrt{\frac{2}{\pi}} e^{2at} D_{-1}(2\sqrt{2at})$$

$$\doteqdot \sqrt{\frac{2}{\pi}} \sum_{v=0}^{\infty} (-1)^v (2at)^{\frac{v}{2}} e^{\frac{1}{2}at} D_{-v-1}(\sqrt{2at})$$

$$\therefore \sum_{v=0}^{\infty} (-1)^v (2at)^{\frac{v}{2}} D_{-v-1}(\sqrt{2at}) = e^{\frac{3}{2}at} D_{-1}(2\sqrt{2at}). \tag{2.4}$$

Similarly the following results have been obtained.

(ii) $$\sqrt{p}\, D_{-v-\frac{1}{2}}(\sqrt{p}\, e^{\frac{i\pi}{4}}) D_{-v-\frac{1}{2}}(\sqrt{p}\, e^{\frac{-i\pi}{4}}) \doteqdot (1+4t^2)^{-\frac{1}{4}} P^{-v}_{-\frac{1}{2}}\left(\frac{1}{\sqrt{1+4t^2}}\right), \tag{2.5}$$

$$R(v) > -\frac{1}{2}.$$

Also $$\sum_{v=0}^{\infty} \varepsilon_{2v}\, p^{-2v}_{-\frac{1}{2}}\left(\frac{1}{\sqrt{1+4t^2}}\right) = \sqrt{\pi}\,(1+4t^2)^{\frac{1}{4}} \tag{2.6}$$

If we take $t=\frac{1}{2}\tan\theta$, we have

$$\sum_{v=0}^{\infty} \varepsilon_{2v}\, P^{-2v}_{-\frac{1}{2}}(\cos\theta) = \sqrt{\pi\sec\theta}. \tag{2.7}$$

(iii) $$\sqrt{p}\, V_v(2\sqrt{p}, 0) \doteqdot \frac{\sin(v\pi)\Gamma\left(\frac{v}{2}\right)}{2\pi^{\frac{3}{2}}\sqrt{t}} e^{\frac{1}{4t}} \Gamma\left(1-\frac{v}{2}, \frac{1}{4t}\right), \tag{2.8}$$

$$R(v) > -1.$$

(iv) $\sqrt{p}\, J_v(\sqrt{2a}\, p^{\frac{1}{4}}) K_v(\sqrt{2a}\, p^{\frac{1}{4}}) \doteqdot \frac{1}{2v\sqrt{\pi t}}\, {}_0F_2\left(1-\frac{v}{2},\ \frac{v}{2}+1\,;\ \frac{a^2}{16t}\right)$

$$+\frac{a^v \Gamma\left(-\frac{v}{2}\right) t^{-\frac{v+1}{2}}}{2^{2v+2}\sqrt{\pi}\,\Gamma(v+1)}\, {}_0F_2\left(1+\frac{v}{2},\ v+1\,;\ \frac{a^2}{16t}\right), \tag{2.9}$$

$$0>R(v)>-1.$$

(b) If $\phi(p) \doteqdot f(t)$, then

$$f(t^2) \doteqdot \frac{p}{\sqrt{\pi}} \int_0^\infty x^{-2} e^{\frac{-p^2}{4x^2}} \phi(x^2)\, dx. \tag{2.10}$$

(i) Consider $f(t)=t^{v-1} e^{-\frac{a^2}{t}} \doteqdot 2a^v p^{1-\frac{v}{2}} K_v(2a\sqrt{p}) \equiv \phi(p)$.

Therefore from (2.10), we obtain

$$t^{2v-2} e^{-\frac{a^2}{t^2}} \doteqdot \frac{2^{v-2} a^v}{\sqrt{\pi}\, p^{v-2}}\, G_{30}^{03}\left(\frac{4}{a^2p^2}\,\middle|\, \frac{3-v}{2},\ 1+\frac{v}{2},\ 1-\frac{v}{2}\right)$$

or, $$t^{2v} e^{-\frac{a^2}{t^2}} \doteqdot \frac{2^{v-1} a^{v+1}}{\sqrt{\pi}\, p^{v-1}}\, G_{30}^{03}\left(\frac{4}{a^2p^2}\,\middle|\, 1-\frac{v}{2},\ \frac{3}{2}+\frac{v}{2},\ \frac{1}{2}-\frac{v}{2}\right) \tag{2.11}$$

Writing $\frac{v}{2}$ for v and summing with respect to v from 0 to ∞, after multiplying by $\frac{(-1)^v}{\Gamma(v+1)}$, we get

$$e^{-t-\frac{a^2}{t^2}} \doteqdot \frac{ap}{2\sqrt{\pi}} \sum_{v=0}^{\infty} \frac{(-1)^v}{\Gamma(v+1)} \left(\frac{2a}{p}\right)^{\frac{v}{2}} G_{30}^{03}\left(\frac{4}{a^2p^2}\,\middle|\, 1-\frac{v}{4},\ \frac{3}{2}+\frac{v}{4},\ \frac{1}{2}-\frac{v}{4}\right).$$

But $$e^{-\frac{a^2}{t^2}} \doteqdot \frac{pa}{2\sqrt{\pi}}\, G_{30}^{03}\left(\frac{4}{a^2p^2}\,\middle|\, 1,\ \frac{3}{2},\ \frac{1}{2}\right).$$

or, $$e^{-t-\frac{a^2}{t^2}} \doteqdot \frac{pa}{2\sqrt{\pi}}\, G_{30}^{03}\left(\frac{4}{a^2(p+1)^2}\,\middle|\, 1, \frac{3}{2},\ \frac{1}{2}\right)$$

Therefore, we can write

$$\sum_{v=0}^{\infty} \frac{(-1)^v}{\Gamma(v+1)} \left(\frac{2a}{p}\right)^{\frac{v}{2}} G_{30}^{03}\left(\frac{4}{a^2p^2}\,\middle|\, 1-\frac{v}{4},\ \frac{3}{2}+\frac{v}{4},\ \frac{1}{2}-\frac{v}{4}\right)$$

$$=G_{30}^{03}\left(\frac{4}{a^2(p+1)^2}\,\middle|\, 1,\ \frac{3}{2},\ \frac{1}{2}\right). \tag{2.12}$$

Similarly the following results can be obtained.

(ii) $t^{2v}J_v(t^2) \doteqdot \frac{2^{v-\frac{7}{2}}p^2}{\pi^{\frac{3}{2}}} G_{13}^{31}\left(\frac{p^4}{64}\middle|\begin{matrix}\frac{1}{2}-v\\ -\frac{1}{4}, \frac{1}{4}, 0\end{matrix}\right)$ (2.13)

In particular, if $v=0$, then we obain

$$J_0(t^2) \doteqdot \frac{\pi p^2}{16} H_{\frac{1}{4}}^{(1)}(p^2/8)H_{\frac{1}{4}}^{(2)}(p^2/8). \tag{2.14}$$

(iii) $t^{-2v}e^{-\frac{a^2}{8t^2}}D_{+2v-1}\left(\frac{a}{\sqrt{2}t}\right) \doteqdot \frac{2^{3v-5/2}p}{\sqrt{\pi}\,a^{2v-1}} G_{03}^{30}\left(\frac{a^2p^2}{16}\middle|0,\ v-\frac{1}{2},\ v\right)$. (2.15)

(iv) $t^{-2k}e^{-\frac{a^2}{2t^2}}W_{k,\mu}(a^2/t^2) \doteqdot \frac{ap^{2k+1}}{2^{2k+1}\sqrt{\pi}} G_{30}^{03}\left(\frac{4}{a^2p^2}\middle|1+k,\ 1+\mu,\ 1-\mu\right)$. (2.16)

3. Rule. If $F(p) \doteqdot f(x)$, $-\infty<R(p)<\infty$, then

$$\frac{\sqrt{pq}}{\sqrt{p}+\sqrt{q}}F(\sqrt{p}+\sqrt{q}) \doteqdot \frac{U(u)U(v)}{\pi\sqrt{uv}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{4}\left(\frac{1}{u}+\frac{1}{v}\right)}f(t)dt \tag{3.1}$$

consider $t^{\lambda-1}e^{-t}U(t) \doteqdot \frac{\Gamma(\lambda)p}{(1+p)^{\lambda}}$, $-1<R(p)<\infty$, $R(\lambda)>0$.

Hence from (3.1), we obtain

$$\frac{\sqrt{pq}}{(\sqrt{p}+\sqrt{q}+1)^{\lambda}} \doteqdot \frac{2^{\frac{\lambda}{2}}(uv)^{\frac{\lambda-1}{2}}}{\pi(u+v)^{\frac{\lambda}{2}}}e^{\frac{uv}{2(u+v)}}D_{-\lambda}\left(\sqrt{\frac{2uv}{u+v}}\right)U(u)U(v), \tag{3.2}$$

$$R(\lambda)>0,\ R(p,\ q)>0.$$

In particular, if $\lambda=2$, we get

$$\frac{\sqrt{pq}}{(\sqrt{p}+\sqrt{q}+1)^2} \doteqdot \left[\frac{2\sqrt{uv}}{\pi(u+v)} - \frac{2uv}{\sqrt{\pi}(u+v)^{\frac{3}{2}}}e^{\frac{uv}{u+v}}\,\mathrm{Erge}\left(\sqrt{\frac{uv}{u+v}}\right)\right]U(u)U(v). \tag{3.3}$$

Similarly the following results in two variables have been obtained.

$$\frac{\sqrt{pq}}{\sqrt{p}+\sqrt{q}}[\log(\sqrt{p}+\sqrt{q})+c] \doteqdot \frac{1}{2\sqrt{\pi(u+v)}}\log\left[\frac{c(u+v)}{uv}\right]U(u)U(v), \tag{3.4}$$

where c is Euler's constant. $R(p,\ q)>0$.

$$\frac{\sqrt{pq}\sin\left[\lambda\tan^{-1}\frac{b}{\sqrt{p}+\sqrt{q}+a}\right]}{[(\sqrt{p}+\sqrt{q}+a)^2+b^2]^{\frac{\lambda}{2}}} \doteqdot \frac{i2^{\frac{\lambda}{2}-1}(uv)^{\frac{\lambda-1}{2}}}{\pi(u+v)^{\frac{\lambda}{2}}}e^{\frac{(a^2-b^2)uv}{2(u+v)}}$$

$$\cdot e^{\frac{iabuv}{u+v}} D_{-\lambda}\Big[(a+ib)\sqrt{\frac{2uv}{u+v}}\Big] - e^{-\frac{iabuv}{u+v}} D_{-\lambda}\Big[(a-ib)\sqrt{\frac{2uv}{u+v}}\Big] U(u)U(v),$$

$$R(\lambda,\ p,\ q)>0. \tag{3.5}$$

$$\sqrt{pq}\Big\{\frac{1}{2}-c\Big[\frac{(\sqrt{p}+\sqrt{q})^2}{4}\Big]\Big\}+\sqrt{pq}\Big\{\frac{1}{2}-s\Big[\frac{(\sqrt{p}+\sqrt{q})^2}{4}\Big]\Big\}^2$$

$$\doteqdot \frac{\tan^{-1}\Big(\frac{4uv}{u+v}\Big)U(u)U(v)}{\pi^2\sqrt{uv}},\quad R(p,\ q)>0. \tag{3.6}$$

$$\frac{\sqrt{pq}}{[(\sqrt{p}+\sqrt{q})^2+a^2]^{\lambda}}\sin\Big[2\lambda\tan^{-1}\Big(\frac{a}{\sqrt{p}+\sqrt{q}}\Big)\Big] \doteqdot \frac{2^{\lambda-\frac{3}{2}}\sec(\lambda\pi)}{\sqrt{\pi}\,\Gamma(2\lambda)}\cdot\frac{(uv)^{\lambda-\frac{1}{2}}}{(u+v)^{\lambda}}$$

$$\times\exp\Big[-\frac{a^2uv}{2(u+v)}\Big]\Big\{D_{2\lambda-1}\Big[-a\sqrt{\frac{2uv}{u+v}}\Big]-D_{2\lambda-1}\Big[a\sqrt{\frac{2uv}{u+v}}\Big]\Big\}U(u)U(v),$$

$$R(\lambda)>-\frac{1}{2},\quad R(p,\ q)>0. \tag{3.7}$$

$$\frac{\sqrt{pq}}{[(\sqrt{p}+\sqrt{q})^2+a^2]^{\lambda}}\cos\Big[2\lambda\tan^{-1}\Big(\frac{a}{\sqrt{p}+\sqrt{q}}\Big)\Big] \doteqdot \frac{2^{\lambda-\frac{3}{2}}\operatorname{cosec}(\lambda\pi)(uv)^{\lambda-\frac{1}{2}}}{\sqrt{\pi}\,\Gamma(2\lambda)(u+v)^{\lambda}}$$

$$\times\ \exp\Big[-\frac{a^2uv}{2(u+v)}\Big]\Big\{D_{2\lambda-1}\Big[a\sqrt{\frac{2uv}{u+v}}\Big]+D_{2\lambda-1}\Big[-a\sqrt{\frac{2uv}{u+v}}\Big]\Big\}U(u)U(v),$$

$$R(\lambda)>0,\quad R(p,\ q)>0. \tag{3.8}$$

$$\sqrt{pq}\,[(\sqrt{p}+\sqrt{q}+a)^{-2\lambda}+(\sqrt{p}+\sqrt{q}-a)^{-2\lambda}]$$

$$\doteqdot \frac{2^{\lambda}(uv)^{\lambda-\frac{1}{2}}}{(u+v)^{\lambda}}\exp\Big[\frac{a^2uv}{2(u+v)}\Big]\Big\{D_{-2\lambda}\Big(-a\sqrt{\frac{2uv}{u+v}}\Big)+D_{-2\lambda}\Big(a\sqrt{\frac{2uv}{u+v}}\Big)\Big\}U(u)U(v),$$

$$R(\lambda)>0,\quad R(p,q)>0,\quad R(p)>|a|,\quad \sqrt{p}+\sqrt{q}>|a|. \tag{3.9}$$

$$\frac{a^{\lambda}\sqrt{pq}\Big[\sqrt{p}+\sqrt{q}+\sqrt{(\sqrt{p}+\sqrt{q})^2+a^2}\Big]^{-\lambda}}{[(\sqrt{p}+\sqrt{q})^2+a^2]^{\frac{1}{2}}} \doteqdot \frac{\exp\Big[-\frac{a^2uv}{2(u+v)}\Big]}{\sqrt{\pi(u+v)}}I_{\frac{\lambda}{2}}\Big[\frac{a^2uv}{2(u+v)}\Big]$$

$$\times U(u)U(v),\quad R(p,\ q)>0,\quad R(\lambda)>-1. \tag{3.10}$$

$$\frac{\sqrt{pq}(\sqrt{p}+\sqrt{q})a^{\lambda}}{\Big[(\sqrt{p}+\sqrt{q})^2+a^2\Big]^{\lambda+\frac{3}{2}}} \doteqdot \frac{(uv)^{\lambda+\frac{1}{2}}e^{-\frac{a^2uv}{u+v}}}{\sqrt{\pi}\,\Gamma(\lambda+3/2)(u+v)^{\lambda+1}}U(u)U(v),$$

$$R(\lambda)>-1,\quad R(p,\ q)>0. \tag{3.11}$$

$$\frac{a^{\lambda}\cot(\lambda\pi)\sqrt{pq}}{\sqrt{(\sqrt{q}+\sqrt{q})^2+a^2}\left[\sqrt{p}+\sqrt{q}+\sqrt{(\sqrt{p}+\sqrt{q})^2+a^2}\right]^{\lambda}}$$

$$-\frac{\sqrt{pq}\left[\sqrt{p}+\sqrt{q}+\sqrt{(\sqrt{p}+\sqrt{q})^2+a^2}\right]^{\lambda}}{a^{\lambda}\sin\lambda\pi\sqrt{(\sqrt{p}+\sqrt{q})^2+a^2}} \doteqdot \frac{\exp\left[-\frac{a^2uv}{2(u+v)}\right]}{\sqrt{\pi(u+v)}}$$

$$\cdot\left\{\tan\left(\frac{\lambda\pi}{2}\right)I_{\frac{\lambda}{2}}\left(\frac{a^2uv}{2(u+v)}\right)+\frac{\sec\left(\frac{\lambda\pi}{2}\right)}{\pi}K_{\frac{\lambda}{2}}\left(\frac{a^2uv}{2(u+v)}\right)\right\}U(u)U(v),$$

$$R(\lambda)>-1,\ a>0,\ R(p,\ q)>0. \tag{3.12}$$

$$\frac{\sqrt{pq}}{[(\sqrt{p}+\sqrt{q})^2+a^2]^{\mu+\frac{1}{2}}}P_{2\mu}^{-2\eta}\left[\frac{\sqrt{p}+\sqrt{q}}{\sqrt{(\sqrt{p}+\sqrt{q})^2+a^2}}\right]$$

$$\doteqdot\frac{2^{2\mu}\Gamma\left(\mu+\eta+\frac{1}{2}\right)}{a\pi\Gamma(2\eta+1)\Gamma(2\mu+2\eta+1)}\cdot\frac{(uv)^{\mu-\frac{1}{2}}}{(u+v)^{\mu}}\exp\left(-\frac{a^2uv}{2(u+v)}\right)$$

$$\cdot M_{\mu,\eta}\left(\frac{a^2uv}{u+v}\right)U(u)U(v),\quad R(\mu+\eta)>-\frac{1}{2}. \tag{3.13}$$

$$\frac{\sqrt{pq}}{[(\sqrt{p}+\sqrt{q})^2+a^2]^{\mu+\frac{1}{2}}}\left\{\frac{\Gamma(2\mu+2\eta+1)}{\tan(2\eta\pi)}P_{2\mu}^{-2\eta}\left(\frac{\sqrt{p}+\sqrt{q}}{\sqrt{(\sqrt{p}+\sqrt{q})^2+a^2}}\right)\right.$$

$$\left.-\frac{\Gamma(2\mu-2\eta+1)}{\sin 2\eta\pi}\cdot P_{2\mu}^{2\eta}\left(\frac{\sqrt{p}+\sqrt{q}}{\sqrt{(\sqrt{p}+\sqrt{q})^2+a^2}}\right)\right\}\doteqdot\frac{2^{2\mu}(uv)^{\mu-\frac{1}{2}}}{a\pi(u+v)^{\mu}}$$

$$\cdot\exp\left(-\frac{a^2uv}{2(u+v)}\right)\times\left\{\frac{\Gamma\left(\mu+\eta+\frac{1}{2}\right)}{\Gamma(2\eta+1)\cot[(\mu-\eta)\pi]}M_{\mu,\eta}\left(\frac{a^2uv}{u+v}\right)-\sec(u-v)\pi\right.$$

$$\left.\cdot W_{\mu,\eta}\left(\frac{a^2uv}{u+v}\right)\right\}U(u)\cdot U(v),\quad R(\mu\pm\eta)>-\frac{1}{2},\ R(p,\ q)>0. \tag{3.14}$$

$$\frac{\sqrt{pq}}{[(\sqrt{p}+\sqrt{q})^2+a^2]^{\frac{\mu+1}{2}}}Q_{\mu}^{y}\left(\frac{\sqrt{p}+\sqrt{q}}{\sqrt{(\sqrt{p}+\sqrt{q})^2+a^2}}\right)\doteqdot\frac{2^{\mu-1}\Gamma\left(\frac{\mu+\eta+1}{2}\right)\Gamma\left(\frac{\mu-\eta+1}{2}\right)}{\pi a\Gamma(\mu-\eta+1)\sin(\mu\pi)}$$

$$\cdot\sin[(\mu+\eta)\pi]\frac{(uv)^{\frac{\mu-1}{2}}}{(u+v)^{\frac{\mu}{2}}}\exp\left(\frac{a^2uv}{2(u+v)}\right)W_{-\frac{\mu}{2},\ \frac{\eta}{2}}\left(\frac{a^2uv}{u+v}\right)U(u)U(v),$$

$$R(\sqrt{p}+\sqrt{q})>a>0,\ R(\mu\pm\eta)>-1. \tag{3.15}$$

$$\sqrt{pq}(\sqrt{p}+\sqrt{q})^{k-\mu-1}\exp\Big[\frac{(\sqrt{p}+\sqrt{q})^2}{8}\Big]W_{-\frac{k+3\mu}{2},\ \frac{k-\mu}{2}}\Big[\frac{(\sqrt{p}+\sqrt{q})^2}{4}\Big]$$

$$\doteqdot\frac{2^{3\mu+k-1}\Gamma\left(2\mu+\frac{1}{2}\right)}{\pi\sqrt{uv}\,\Gamma(4\mu+1)}\Big[1+\frac{u+v}{4uv}\Big]^{-2\mu-\frac{1}{2}}$$

$$\cdot{}_2F_1\Big[2\mu+\frac{1}{2},\ \mu-k+\frac{1}{2}\ ;\ 2\mu+1\ ;\ \frac{4uv}{u+v+4uv}\Big]U(u)U(v),$$

$$R(\mu)>-\frac{1}{4},\ R(p,\ q)>0. \tag{3.16}$$

4. Consider

$$\int_0^\infty t^{\lambda-1}e^{-2\sqrt{at}}e^{-pt}dt=\frac{\Gamma(2\lambda)}{2^{\lambda-1}}p^{-\lambda}e^{\frac{a}{2p}}D_{-2\lambda}\Big[\sqrt{\frac{2a}{p}}\Big],$$

or,

$$\int_0^\infty t^{2\lambda-1}e^{-2\sqrt{a}\,t}e^{-pt^2}dt=\frac{\Gamma(2\lambda)}{2^{\lambda}}p^{-\lambda}e^{\frac{a}{2p}}D_{-2\lambda}\Big(\sqrt{\frac{2a}{p}}\Big). \tag{4.1}$$

Writing $(pq)^{-\frac{1}{2}}$ for p and multiplying both the sides of (4.1) by $p^{-\frac{1}{2}}(pq)^{1-v}$ and then interpreting, we get

$$(\pi y)^{-\frac{1}{2}}(4\times y)^{\frac{v}{2}-\frac{1}{4}}\int_0^\infty t^{2\lambda-2v}e^{-2\sqrt{a}\,t}J_{2v-1}[(64\times y)^{\frac{1}{4}}t]\,dt$$

$$\doteqdot\frac{\Gamma(2\lambda)}{2^{\lambda}}p^{-\frac{1}{2}}(pq)^{1-v+\frac{\lambda}{2}}e^{\frac{a}{2}\sqrt{pq}}D_{-2\lambda}[\sqrt{2a}(pq)^{\frac{1}{4}}] \tag{4.2}$$

or,

$$p^{-\frac{1}{2}}(pq)^{1-v+\frac{\lambda}{2}}e^{\sqrt{apq}}D_{-2\lambda}(2\sqrt{2}(apq)^{\frac{1}{4}})\doteqdot\frac{y^{-\frac{1}{2}}(xy)^{\frac{v}{2}-\frac{1}{4}}}{\sqrt{\pi}\,2^{2(\lambda-v+1)}(\sqrt{a}+\sqrt{xy})^{\lambda-v+\frac{1}{2}}}$$

$$\cdot P^{1-2v}_{2\lambda-2v}\Big[\frac{a^{\frac{1}{4}}}{\sqrt{\sqrt{a}+\sqrt{xy}}}\Big],\ R(\lambda)>0. \tag{4.3}$$

By considering the following integrals:

$$\int_0^\infty e^{-pt}\frac{t^{\lambda}}{t+a}dt=\Gamma(\lambda+1)a^{v}e^{ap}\Gamma(-v,\ ap),$$

$$\int_0^\infty e^{-\left(p+\frac{1}{a}\right)t}t^{\lambda-1}dt=\frac{\Gamma(\lambda)}{\left(p+\frac{1}{a}\right)^{\lambda}},$$

$$\int_0^\infty e^{-pt} t^{v-\frac{3}{4}}(t+2)^{\frac{1}{4}-v} p_\mu^{\frac{1}{2}-2v}(1+t)dt$$

$$=\frac{p^{-\frac{\mu}{2}-v-\frac{1}{4}} 2^{\frac{1}{2}-2v}}{\Gamma\left(\mu+2v+\frac{1}{2}\right)\Gamma\left(2v-\mu-\frac{1}{2}\right)} E\left(\mu+2v+\frac{1}{2},\ 2v-\mu-\frac{1}{2},\ \frac{\mu}{2}+v+\frac{1}{4}:\ 2v+\frac{1}{2};\ 2p\right),$$

$$\int_0^\infty e^{-pt} t^{\mu-\frac{1}{2}} J_{2v}(2\sqrt{at})dt=\frac{\Gamma\left(\mu+v+\frac{1}{2}\right)}{\sqrt{a}\,\Gamma(v+21)} p^{-\mu} e^{-\frac{a}{2p}} M_{\mu,v}(a/p),$$

$$\int_0^\infty e^{-(p+b)t} t^{\lambda-1} J_{2\mu}(2\sqrt{at}) J_{2\omega}(2\sqrt{at})dt$$

$$=\frac{2\Gamma(\lambda+\mu+\omega)a^{\mu+\omega}}{\Gamma(2\mu+1)\Gamma(2\omega+1)(p+b)^{\lambda+\mu+\omega}}\ {}_3F_3\left[\begin{matrix}\mu+\omega+\frac{1}{2},\ \mu+\omega+1,\ \lambda+\mu+\omega: \\ 2\mu+1,\ 2\omega+1,\ 2\mu+2\omega+1;\end{matrix}\ -\frac{4a}{b+p}\right],$$

$$\int_0^\infty e^{-pt} t^{2v-1}\cos(at)dt=\frac{\Gamma(2v)}{(p^2+a^2)^v}\cos\left[2v\tan^{-1}(a/p)\right],$$

$$\int_0^\infty e^{-\left(p+\frac{1}{4}\right)t} t^{\lambda-1} M_{k,\mu}(t/2)dt=\frac{2^{-\mu-\frac{1}{2}}\Gamma\left(\mu+\lambda+\frac{1}{2}\right)}{\left(p+\frac{1}{2}\right)^{\mu+\lambda+\frac{1}{2}}}\ {}_2F_1\left[\begin{matrix}\mu+\lambda+\frac{1}{2},\ \mu-k+\frac{1}{2}, \\ 2\mu+1:\ \frac{1}{2p+1}\end{matrix}\right],$$

$$\int_0^\infty e^{-\left(p+\frac{a}{2}\right)t} t^{\lambda-1} W_{k,\mu}(at)dt=\frac{\Gamma\left(\mu+\lambda+\frac{1}{2}\right)\Gamma\left(\lambda-\mu+\frac{1}{2}\right)a^{\mu+\frac{1}{2}}}{\Gamma(\lambda-k+1)(p+a)^{\lambda+\mu+\frac{1}{2}}}$$

$$\cdot {}_2F_1\left(\lambda+\mu+\frac{1}{2},\ \mu-k+\frac{1}{2};\ \lambda-k+1;\ \frac{p}{p+a}\right),$$

$$\int_0^\infty e^{-pt-\frac{a}{4}t} t^{v-\frac{1}{2}} J_{v-\frac{1}{2}}\left(\frac{b}{4}t\right)dt=\frac{\Gamma(2v)}{\left[\left(p+\frac{a}{4}\right)^2+\frac{b^2}{16}\right]^{\frac{v}{2}+\frac{1}{4}}}\cdot P_{v-\frac{1}{2}}^{\frac{1}{2}-v}\left(\frac{p+\frac{a}{4}}{\sqrt{\left(p+\frac{a}{4}\right)^2+\frac{b^2}{16}}}\right),$$

and

$$\int_0^\infty e^{-pt} t^{v-\frac{1}{2}} \eta_{v-\frac{1}{2}}(at)dt=\frac{\Gamma(2v)\cot\left[\left(v-\frac{1}{2}\right)\pi\right]}{(p^2+a^2)^{\frac{v}{2}+\frac{1}{4}}} P_{v-\frac{1}{2}}^{\frac{1}{2}-v}\left(\frac{p}{\sqrt{p^2+a^2}}\right)$$

$$-\operatorname{cosec}\left[\left(v-\frac{1}{2}\right)\pi\right] P^{\frac{1}{2}-v}_{v-\frac{1}{2}}\left(\frac{p}{\sqrt{p^2+a^2}}\right) ;$$

we arrive at the following results.

$$p^{-\frac{1}{2}}(pq)^{1-v}\exp\left(\frac{a}{\sqrt{pq}}\right)\Gamma\left(-v,\ \frac{a}{\sqrt{pq}}\right) \fallingdotseq \frac{2^{2v-\frac{1}{4}}a^{\lambda-v+\frac{1}{4}}\Gamma\left(\lambda+\frac{5}{4}\right)\Gamma\left(-\lambda-\frac{1}{4}\right)}{\pi^{\frac{3}{2}}\Gamma(2v)\Gamma(\lambda+1)}$$

$$\cdot x^{v-\frac{3}{8}}y^{v-\frac{7}{8}}{}_0F_1(2v\ ;\ 2a\sqrt{xy}) + \frac{2^{-v-\lambda-\frac{1}{2}}\Gamma\left(\lambda+\frac{3}{4}\right)x^{\frac{v}{2}-\frac{\lambda}{2}-\frac{1}{2}}}{a^v\Gamma(\lambda+1)\Gamma\left(2v-\lambda-\frac{5}{2}\right)\sqrt{\pi}}$$

$$\cdot y^{-\frac{v}{2}-\frac{\lambda}{2}-1}{}_1F_2\left(1\ ;\ 2v-\lambda-\frac{5}{2},\quad -2\lambda-\frac{1}{2}\ ;\ 2a\sqrt{xy}\right),$$

$$-\frac{1}{4}>\lambda>-\frac{3}{4}. \tag{4.4}$$

$$\sqrt{\frac{\pi}{p}}\ \frac{(pq)^{\frac{\lambda}{2}-v+1}}{(a+\sqrt{pq})^{\lambda}} \fallingdotseq \frac{2^{v+2}}{\Gamma(2v)a^v}x^{\frac{v-1}{2}}y^{\frac{v}{2}-1}e^{-a\sqrt{xy}}M_{\lambda-v,\,v-\frac{1}{2}}(2a\sqrt{xy}),$$

$$R(\nu)>0. \tag{4.5}$$

$$p^{-\frac{1}{2}}(pq)^{\frac{\mu-v}{2}+\frac{5}{4}}E\left(\mu+2v+\frac{1}{2},\ 2v-\mu-\frac{1}{2},\ \frac{\mu}{2}+v+\frac{1}{4}\ ;\ 2v+\frac{1}{2}\ ;\ \frac{2}{\sqrt{pq}}\right)$$

$$\fallingdotseq \frac{2}{\pi}\sqrt{\frac{2}{y}}\,(4\times y)^{\frac{v}{2}-\frac{3}{8}}\left\{K_{\mu+\frac{1}{2}}[\sqrt{2}(4\times y)^{\frac{1}{4}}]\right\}^2. \tag{4.6}$$

$$\frac{b}{\sqrt{a}}\cdot\frac{p^{-\frac{1}{2}}(pq)^{1-v}}{[b+(pq)^{-\frac{1}{2}}]^v}\exp\left[-\frac{a}{2\left(b+\frac{1}{\sqrt{pq}}\right)}\right]M_{u,v}\left(\frac{a}{b+\frac{1}{\sqrt{pq}}}\right)$$

$$\fallingdotseq(\pi y)^{-\frac{1}{2}}(4\times y)^{\frac{v}{2}-\frac{1}{4}}\exp\left(-\frac{a+2\sqrt{xy}}{b}\right)I_v\left[\frac{2\sqrt{a}}{b}(4\times y)^{\frac{1}{4}}\right]. \tag{4.7}$$

$$(\pi y)^{-\frac{1}{2}}(4\times y)^{v-\frac{1}{2}}\phi_2\left(\lambda+\omega+\mu+v-\frac{1}{2}\ ;\ 2\mu+1,\ 2\omega+1,\ 2v\ ;\ ab,\ 1,\ 2b\sqrt{xy}\right)$$

$$\fallingdotseq\frac{2\Gamma(\lambda+\mu+\omega)\Gamma(2v)}{\Gamma\left(\lambda+\mu+\omega+v-\frac{1}{2}\right)}\cdot\frac{p^{-\frac{1}{2}}(pq)^{\frac{\lambda+\mu+\omega}{2}-v+1}b^{\lambda+\mu+\omega}}{\left(b+\sqrt{pq}\right)^{\lambda+\mu+\omega}}$$

$$\cdot{}_3F_3\left[\begin{matrix}\mu+\omega+\frac{1}{2},\ \mu+\omega+1,\ \lambda+\mu+\omega\ ; \\ 2\mu+1,\ 2\omega+1,\ 2\mu+2\omega+1\ ;\end{matrix}\ -\frac{4ab\sqrt{pq}}{b+\sqrt{pq}}\right] \tag{4.8}$$

$$(\pi y)^{-\frac{1}{2}}(4\times y)^{v-\frac{1}{2}}\cos\left(v\pi-\frac{2}{a}\sqrt{xy}\right)$$

$$\doteqdot\frac{\Gamma(2v)a^{2v}p^{-\frac{1}{2}}(pq)^{1-v}}{\left(a^2+\frac{1}{pq}\right)^{v}}\cos\left[2v\tan^{-1}(a\sqrt{pq})\right]. \qquad (4.9)$$

$$(\pi y)^{-\frac{1}{2}}(4\times y)^{\frac{v}{2}+\frac{k}{4}+\frac{\mu}{4}-\frac{5}{8}}\exp(-2\sqrt{xy})W_{\alpha,\beta}(4\sqrt{xy})$$

$$\doteqdot\frac{2^{-\frac{k+\mu+v}{2}-\frac{3}{4}}\Gamma\left(\frac{3v+1}{2}\right)\Gamma\left(\mu+k+\frac{1}{2}\right)}{\Gamma(2\mu+1)\sqrt{p}\,(2+\sqrt{pq})^{\frac{3v+1}{2}}}(pq)^{\frac{v}{2}+\frac{5}{4}}$$

$$\cdot{}_2F_1\left(\begin{matrix}\frac{3v+1}{2},\ \mu-k+\frac{1}{2}:\\ 2\mu+1:\end{matrix}\ \frac{\sqrt{pq}}{2+\sqrt{pq}}\right); \qquad (4.10)$$

where $2\alpha=k-3\mu+2\nu-\frac{1}{2}$, $2\beta=k+\mu-2\nu$.

$$(\pi y)^{-\frac{1}{2}}(4\times y)^{v-\frac{3}{8}}{}_2F_2\left(\begin{matrix}\lambda+\mu+\frac{1}{2},\ \lambda-\mu+\frac{1}{2}:\\ 2v,\ 1-k+\lambda:\end{matrix}\ -\frac{2}{a}\sqrt{xy}\right)$$

$$\doteqdot\frac{a^{\mu-\lambda+\frac{1}{2}}\Gamma(2v)(pq)^{1-v}}{\sqrt{2p}\left(a+\frac{1}{\sqrt{pq}}\right)^{\lambda+\mu+\frac{1}{2}}}{}_2F_1\left(\begin{matrix}\lambda+\mu+\frac{1}{2},\ \mu-k+\frac{1}{2}:\\ \lambda-k+1:\end{matrix}\ \frac{1}{1+a\sqrt{pq}}\right),$$

$$R(\nu)>0. \qquad (4.11)$$

$$\frac{4(\pi y)^{-\frac{1}{2}}(4\times y)^{\frac{v}{2}-\frac{1}{4}}}{\sqrt{a^2+b^2}}\exp\left[-\frac{8a\sqrt{xy}}{a^2+b^2}\right]J_{v-\frac{1}{2}}\left(\frac{8b\sqrt{xy}}{a^2+b^2}\right)$$

$$\doteqdot\frac{\Gamma(2v)p^{-\frac{1}{2}}(pq)^{1-v}}{\left[\left(\frac{1}{\sqrt{pq}}+\frac{a}{4}\right)^2+\frac{b^2}{16}\right]}P^{\frac{1}{2}-v}_{v-\frac{1}{2}}\left[\frac{\frac{1}{\sqrt{pq}}+\frac{a}{4}}{\sqrt{\left(\frac{1}{\sqrt{pq}}+\frac{a}{4}\right)^2+\frac{b^2}{16}}}\right],$$

$$R(\nu)>0. \qquad (4.12)$$

$$(\pi y)^{-\frac{1}{2}}(4\times y)^{\frac{v}{2}-\frac{1}{4}}H_{v-\frac{1}{2}}(2a\sqrt{xy})$$

$$\doteqdot\frac{\operatorname{cosec}\left(v-\frac{1}{2}\right)\pi}{a\sqrt{p}\,(pq)^{v-1}}p^{\frac{1}{2}-v}_{v-\frac{1}{2}}\left[\frac{a}{\sqrt{a^2+pq}}\right]-\frac{\Gamma(2v)\cot\left(v-\frac{1}{2}\right)\pi a^{v-\frac{1}{2}}}{(a^2+pq)^{\frac{v}{2}+\frac{1}{4}}\sqrt{p}\,(pq)^{\frac{v}{2}-\frac{5}{4}}}$$

$$\cdot p^{\frac{1}{2}-v}{}_{v-\frac{1}{2}}\left[\frac{a}{\sqrt{a^2+pq}}\right]. \tag{4.13}$$

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