SUBSTITUTION FORMULAE FOR MEIJER TRANSFORM

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1. Meijer integral transforms are of great value in solving differential equations of the Bessel type.

They are defined by the integral

(1, 1) $$\phi(p)=\int_0^{\infty}\sqrt{px}\ k_{\nu}(px)\ f(x)\ dx,$$

where $k_{\nu}(x)$ is Macdonald's function. The inversion formula takes the form

(1.2) $$f(x)=\frac{1}{2\pi i}\lim_{\lambda\to\infty}\int_{c-i\lambda}^{c+i\lambda}I_{\nu}(px)\sqrt{px}\ \phi(p)dp.$$

Many formulae have been calculated by different authors. In the present paper, I have obtained a theorem by using the property of Laplace and Mellin transformations. I have used this theorem to evaluate some integrals involving Meijer transform, not easy to tackle otherwise, in a neat form. The results are given in the form of a table, which are believed to be new.

Through out this paper the notations given in integral transforms and operational calculus by Prudnikov & Ditkin have been followed.

2. THEOREM 1. *Let*

$$\text{(i)}\ \phi(p)\doteqdot f(t^{-n})$$

$$\text{(ii)}\ p^{\mu-\frac{1}{n}}f(p)\doteqdot g(x),$$

then

(2.1) $$\frac{\phi(p)}{p^{n(1-\mu)+1}}=\frac{n^{n(\mu-1)-\frac{1}{2}}}{(2\pi)^{\frac{n-1}{2}}}\int_0^{\infty} g(x)\ G_{0,n+1}^{n+1,0}\left(\frac{p^n x}{n^n}\,\Big|\,0,\mu-1,\mu-1+\frac{1}{n},\cdots,\mu-\frac{1}{n}\right)dx,$$

provided $g(x)$ *is bounded and absolutely integrable in* $(0,x)$ *or in* $(0,\infty)$.

PROOF. we have from (i, ii)

$$\phi(p)=p\int_0^{\infty}e^{-pt}f(t^{-n})dt$$

$$\text{and}\ t^{\mu-\frac{1}{n}-1}f(t)=\int_0^{\infty}e^{-tx}g(x)dx.$$

$$\text{or,}\quad t^{n-n\mu+1}f(t^{-n})=\int_0^\infty e^{-t^{-1}x}g(x)dx.$$

(2.2) $$\therefore\ \phi(p)=p\int_0^\infty t^{n\mu-n-1}e^{-pt}\,dt\int_0^\infty e^{-t^{-1}x}g(x)dx.$$

On changing the order of integration, which is justifiable, we have

(2.3) $$\phi(p)=p\int_0^\infty g(x)dx\int_0^\infty t^{n\mu-n-1}\,e^{-pt-t^{-1}x}dt.$$

(2.4) $$\text{Let } I=\int_0^\infty t^{n\mu-n-1}\,e^{-pt-t^{-1}x}dt.$$

Apply the method of Mellin transform to solve the above integral (2.4).

$$\therefore\ I=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(n(1-\mu)-s)}{p^{n(\mu-1)-s}}\cdot\frac{x^{\frac{s}{n}}}{n}\Gamma(-s/n)ds$$

$$\text{or,}\quad I=\frac{p^{n(1-\mu)}n^{n(\mu-1)-\frac{1}{2}}}{2\pi i\ (2\pi i)^{\frac{n-1}{2}}}\int_{C_1}\Gamma(s)\ \Gamma(\mu-1+s)\ \Gamma(\mu-1+s+\frac{1}{n})\cdots\cdots$$

$$\cdots\cdots\Gamma\left(\mu-1+s+\frac{n-1}{n}\right)\frac{n^{ns}}{p^{ns}x^{s}}ds$$

$$\text{or,}\quad I=\frac{p^{n(1-\mu)}n^{n(\mu-1)-\frac{1}{2}}}{2\pi i\ (2\pi i)^{\frac{n-1}{2}}}\int_{C_2}\frac{p^{ns}x^{s}}{n^{sn}}\cdot\Gamma(-s)\ \Gamma(\mu-1-s)\ \Gamma\left(\mu-1-s+\frac{1}{n}\right)\cdots\cdots$$

$$\cdots\cdots\Gamma\left(\mu-1-s+\frac{n-1}{n}\right)ds$$

(2.5) $$I=\frac{p^{n(1-\mu)}n^{n(\mu-1)-\frac{1}{2}}}{(2\pi)^{\frac{n-1}{2}}}G^{n+1,0}_{0,n+1}\left(\frac{p^n x}{n^n}\,\middle|\,0,\mu-1,\mu-1+\frac{1}{n},\cdots,\mu-\frac{1}{n}\right).$$

Now using (2.3) and (2.5) to get the desired result (2.1). Thus (2.1) is proved.

3. COROLLARY.

On putting $n=1$, $\mu=m+1$ in (2.1), we get

(3.1) $$p^{\frac{m}{2}}\phi(p)=2p\int_0^\infty x^{\frac{m}{2}}K_m(2\sqrt{px})g(x)dx.$$

4. EXAMPLES BASED ON THE COROLLARY.

$$\text{Let } f(1/t)=\frac{H_1(at)}{t}\doteqdot\frac{2p}{\pi}\left[-1+\frac{\sqrt{p^2+a^2}}{a^2}\log\frac{a+\sqrt{p^2+a^2}}{p}\right]\equiv\phi(p)$$

$$p^m f(p)=p^{m+1}H_1\ (a/p)\doteqdot\frac{2a^2\,t^{1-m}}{3\pi\Gamma(2-m)}{}_1F_4\left(1\ ;\ \frac{3}{2},\frac{5}{2},1-\frac{m}{2},\frac{3-m}{2}\ ;\ -\frac{a^2t^2}{16}\right)\equiv g(a).$$

Hence from the corollary, we get

$$(4.1)\ \int_0^\infty x^{1-\frac{m}{2}} K_m(2\sqrt{px})\,{}_1F_4\left(1\,;\frac{3}{2},\frac{5}{2},1-\frac{m}{2},\frac{3-m}{2}\,;\,-\frac{a^2x^2}{16}\right)dx$$

$$=\frac{3\Gamma(2-m)p^{m/2}}{2a^2}\left[\frac{\sqrt{p^2+a^2}}{a}\log\frac{a+\sqrt{a^2+p^2}}{p}-1\right];\ R(m)<2,\ R(p)>\frac{a}{2}.$$

In a similar manner, the following formulae have been obtained.

No.	$g(x)$	$\phi(p)=2p^{1-\frac{m}{2}}\int_0^\infty x^{\frac{m}{2}}K_m(2\sqrt{px})\times g(x)dx$
(4.2)	$x^{\frac{v}{2}-m}e^{-x}L_{m-\frac{1}{2}}^{\frac{v}{2}-m}(x)$	$\dfrac{2^{\frac{v+1}{2}}\Gamma\left(\frac{v}{2}+1\right)\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)}e^{p/2}D_{-v-1}\times(\sqrt{2p})$, $R(v)>-2,\ R(v-2m)>-2.$
(4.3)	$x^{v-m-1}\ J^{1/2}_{v-m-1}(2\sqrt{ax})$	$\dfrac{\Gamma(2v)e^{a/2p}}{2^{v-1}\ p^{v-1}}D_{-2v}(\sqrt{2a/p}).$ $R(v)>0,\ R(v-m)>0.$
(4.4)	$x^{v-m-1}J^2_{v-m-1}(x^2/8a)$	$\Gamma(v)2^v a^{v/2}p\,e^{ap^2}D_{-v}(2\sqrt{a}p),$ $R(v)>0,\ R(v-m)>0.$
(4.5)	$x^{v-m-\frac{1}{2}}\ {}_0F_2\left(v-m+\frac{1}{2},\frac{3}{2}\,;\,-\frac{ax}{2}\right)$	$\dfrac{\sqrt{\pi}\,\Gamma\left(v-m+\frac{1}{2}\right)\sec(v\pi)}{2^{v+1}\sqrt{a}\ p^{v-1}}e^{-a/4p}$ $\times[D_{2v-1}(-\sqrt{a/p})-D_{2v-1}(\sqrt{a/p})],$ $R(v)>-\frac{1}{2},\ R(v-m)>-\frac{1}{2}.$
(4.6)	$x^{\mu-m}S_2\left(\frac{v-1}{2},\frac{-v-1}{2},\frac{m-\mu}{2},\frac{m-\mu-1}{2}\,;\,\frac{ax}{4}\right)$	$\dfrac{2^{\mu-m+1}\sin(\mu\pi)\Gamma(\mu-v+1)p}{\sqrt{\pi}\sin[(\mu+v)\pi]\ (p^2-a^2)^{\frac{v+1}{2}}}$ $\times Q^v_\mu\left(\frac{p}{\sqrt{p^2-a^2}}\right),\ R(\mu\pm v)>-1,$ $R(\mu\pm v-m)>-1,\ R(p)>\frac{a}{2}.$
(4.7)	$x^{\frac{v-m-1}{2}}[I_{v-m-1}(2\sqrt{ax})-J_{v-m-1}(2\sqrt{ax})]$	$\Gamma(v)a^{\frac{v-m+1}{2}}p[(p-a)^{-v}-(p+a)^{v}],$ $R(v)>0,\ R(v-m)>0,\ R(p)>R\lvert a\rvert.$
(4.8)	$x^{v-m-1}[\phi(v-m)-\log x]$	$-\Gamma(v)\Gamma(v-m)p^{1-v}[\phi(v)-\log p],$ $R(v)>0,\ R(v-m)>0.$

(4, 9)	$bei(2\sqrt{x})$	$\frac{i}{2}\Gamma(m+1)p[(p+i)^{-m-1}$ $-(p-i)^{-m-i}]$, $R(m)>-1,\ R(p)>\frac{1}{2}$.
(4. 10)	$x^{\beta-m}{}_1F_2(-n;\alpha+1,\ \beta-m+1;x)$, $R(\beta)>-1,\ R(\beta-m)>-1$.	$\frac{\Gamma(\beta+n+1)\Gamma(\alpha+1)\Gamma(\beta-m+1)}{\Gamma(\alpha+n+1)\ p^{\beta+n}}$ $\frac{(p-1)^n}{}\times{}_2F_1\left(-n,\alpha-\beta;-\beta-n;\frac{p}{p-1}\right)$.
(4. 11)	$x^{v-m-1}{}_1F_2\left(1;\frac{v-m}{2},\frac{v-m+1}{2};-\frac{x^2}{4}\right)$	$\pi p\ V_v(2p,0)\ \Gamma(v-m)\ \mathrm{cosec}(v\pi)$, $R(v)>0,\ R(v-m)>0$.
(4. 12)	$x^{-m}{}_1F_2\left(\frac{1}{2}-v;\frac{1-m}{2},1-\frac{m}{2};-\frac{x^2}{4}\right)$	$2^{v-1}\sqrt{\pi}\Gamma(v+\frac{1}{2})\Gamma(1-m)p^{1-v}$ $\times[H_v(p)-Y_v(p)]$, $R(m)<1$.
(4. 13)	$x^{1-m}{}_1F_2\left(1;1-\frac{m}{2},\frac{3-m}{2};-\frac{x^2}{4a^2}\right)$	$\Gamma(2-m)a^2p[ci(ap)\cos(ap)-si(ap)$ $\sin(ap)]$. $R(m)<2$.
(4. 14)	$x^{-m}{}_2F_1\left(\frac{1}{2}-K+\mu,\frac{1}{2}-K-\mu;\ 1-m;-\frac{x}{a}\right)$	$2^{1-2K}\ a^{\frac{1}{2}-K}\Gamma(1-m)\ p^{\frac{1}{2}-K}$ $\times S_{2K,2\mu}(2\sqrt{ap})$, $R(m)<1$.
(4. 15)	$x^{v+\mu-m}{}_1F_2\left(v+\frac{1}{2};2v+1,1+\mu-m+v;2ax\right)$	$\frac{\Gamma(1+v)\Gamma(1+\mu-m+v)}{a^v2^{-v}(p+2a)^{\frac{\mu+1}{2}}\ p^{\frac{\mu-1}{2}}}$ $\times P_\mu^{-v}\left(\frac{p+a}{\sqrt{p^2+2ap}}\right)$, $R(v+\mu)>-1$, $R(v+\mu-m)>-1,\ R(p)>\lvert 2a\rvert$.
(4. 16)	$x^{v+\mu-m-\frac{1}{2}}\ {}_0F_2\left(v+\mu-m+\frac{1}{2},\ 2v+1;-ax\right)$	$\frac{\Gamma\left(\mu+v+\frac{1}{2}\right)\Gamma\left(v+\mu-m+\frac{1}{2}\right)}{2^{2v+1}\ a^{v+\frac{1}{2}}\ \ p^{\mu-1}}$ $\times e^{-a/2p}M_{\mu,v}(a/p)$, $R(v+\mu)>-\frac{1}{2}$, $R(v+\mu-m)>-\frac{1}{2}$.

(4.17)	$x^{v+\mu-m}{}_0F_3\left(v+1, \frac{v+\mu-m+1}{2}, \frac{v+\mu-m+2}{2}; -\frac{a^2x^2}{4}\right)$, $R(v+\mu)>-1$, $R(v+\mu-m)>-1$.	$\frac{2^v\Gamma(v+1)\Gamma(1+v+\mu-m)\Gamma(\mu+v+1)}{a^v(p^2+a^2)^{\frac{\mu+1}{2}}} \times p^{m/2}P_\mu^{-v}\left(\frac{p}{\sqrt{p^2+a^2}}\right)$, $R(p)>R(a/2)$.
(4.18)	$x^{\beta-m-1}{}_1F_3(-n; \alpha+1, \beta, \beta-m; \lambda x)$	$\frac{\Gamma(\beta-m)\Gamma(n+1)}{p^{\beta-1}}\left[\frac{\Gamma(\beta)}{(\alpha+1)_n}\right] \times L_n^\alpha\left(\frac{\lambda}{p}\right)$; $R(\beta)>0$, $R(\beta-m)>0$.
(4.19)	$x^{\gamma-m-1}{}_2F_3\left(-n, n+2v; v+\frac{1}{2}, \gamma, \gamma-m; x\right)$	$\frac{\Gamma(\gamma-m)\Gamma(n+1)\Gamma(2v)\Gamma(\gamma)}{\Gamma(n+2v)\ p^{\gamma-1}} \times C_n^v\left(1-\frac{2}{p}\right)$, $R(\gamma)>0$, $R(\gamma-m)>0$.
(4.20)	$x^{-\frac{1}{2}}\left[\sin\left(v+\frac{1}{2}\right)\pi J_{2v+1}(\sqrt{8ax}) + \cos\left(v+\frac{1}{2}\right)\pi Y_{2v+1}(\sqrt{8ax})\right]$	$-\frac{\Gamma(m-v)\Gamma(m+v+1)p^{1-\frac{m}{2}}}{\sqrt{2a}\ (p+2a)^{m/2}} \times P_v^{-m}\left(\frac{p+a}{a}\right)$; $0>R(v)>-1$, $R(m+v)>-1$, $R(m-v)>0$.
(4.21)	$x^{\frac{1}{4}-v}P_{2v-\frac{1}{2}}^{2v-\frac{1}{2}}\left(\sqrt{\frac{a+x}{a}}\right)$	$\frac{\Gamma\left(2v+\frac{1}{4}\right)a^{v+\frac{3}{4}}}{2^{3/2-2v}p^{2v-1}}H_{2v}^{(1)}(\sqrt{ap}) \times H_{2v}^{(2)}(\sqrt{ap})$; $m=4v-\frac{1}{2}$, $\frac{3}{4}>R(v)>-\frac{1}{2}$.
(4.22)	$x^{\frac{v}{2}}\left[I_v(2\sqrt{ax})-L_v(2\sqrt{ax})\right]$	$\frac{a^{\frac{v}{2}}p^{\frac{1}{2}}}{\sqrt{p}+\sqrt{a}}$; $m=-v$, $R(v)>-1$.
(4.23)	$x^{v-m-1}\left[\frac{\Gamma(-2\mu)(ax)^{\mu+\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-K-\mu\right)\left(\Gamma\frac{1}{2}+\mu+v-m\right)} \times {}_1F_2\left(\frac{1}{2}-K+\mu; 1+2\mu, \frac{1}{2}+\mu+v-m; ax\right)+\frac{\Gamma(2\mu)(ax)^{\frac{1}{2}-\mu}}{\Gamma\left(\frac{1}{2}-K+\mu\right)\Gamma\left(\frac{1}{2}-\mu+v-m\right)}\right.$	$\frac{\Gamma\left(\mu+v+\frac{1}{2}\right)\Gamma\left(v-\mu+\frac{1}{2}\right)a^{\mu+\frac{1}{2}}}{\Gamma(v-K+1)\ p^{\mu+v-\frac{1}{2}}} \times {}_2F_1\left(\mu+v+\frac{1}{2}, \mu-K+\frac{1}{2}; v\text{-}K+1; 1-\frac{a}{p}\right)$, $R\left(v\pm\mu+\frac{1}{2}\right)>0$,

	$\times {}_1F_2\left(\frac{1}{2}-K-\mu\,;\,1-2\mu,\ \frac{1}{2}-\mu+v-m;ax\right)\Big]$	$R\left(v-m\pm\mu+\frac{1}{2}\right)>0.$
(4.24)	$(x+a)^{v-1}$	$\dfrac{a^{\frac{\mu}{2}}\Gamma(1+\mu-v)}{2^{\mu-2v-2}p^{\frac{\mu}{2}-1}}S_{2v-\mu-1,\,\mu}(2\sqrt{ap}),$ $m=\mu-v,\ \ R(\mu-v)>-1.$

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REFERENCES

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