USE OF GENERALIZED LEGENDRE ASSOCIATED FUNCTIONS AND THE H-FUNCTION IN HEAT PRODUCTION IN A CYLINDER

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1. Introduction

The object of the present paper is to employ Fox's H-function and generalized Legendre associated functions to solve the fundamental differential equation of the diffusion of heat in a cylinder of radius a when there are sources of heat within it which lead to an axially symmetrical temperature distribution. According to Sneddon [8, p. 202(166)], the fundamental differential equation is of the form

$$(1.1)\qquad \frac{\partial u}{\partial t}=\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\theta(r,t).$$

If we assume that the cylinder is infinitely long and the rate of generation of heat is independent of temperature, then the variation with z may be neglected. In addition, we suppose that the surface $r=a$ is maintained at zero temperature and initial distribution of temperature is also zero.

In particular, we suppose

$$(1.2)\qquad \theta(r,t)=\frac{k}{K}f(r)g(t),$$

where k is the diffusivity and K the conductivity of the material.

Single function $f(r)$ can represent both sources and sinks embedded in the system. Whenever the product $f(r)g(t)$ gives a negative value, it should be treated as a sink. If $g(t)>0$, then the inner circular cylinder will enclose sources, while the volume between two concentric cylinders will contain sinks. If $g(t)<0$, then sources and sinks will interchange their roles.

In [4], Kuipers and Meulenbeld have defined generalized Legendre associated functions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ as two linearly independent solutions of the differential equation

$$(1.3)\qquad (1-z^2)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+\left\{k(k+1)-\frac{m^2}{2(1-z)}-\frac{n^2}{2(1+z)}\right\}w=0,$$

at all points of the z-plane in which a cross-cut exists along the real axis from 1 to $-\infty$ and in [6], these functions have been defined for the real values of z on the cross-cut for $-1<z<1$.

The H-function has been introduced by Fox [3, p. 408] and its conditions of validity, asymptotic expansions and analytic continuations have been discussed by Braaksma [1]. Following the definition given by Braaksma [1, pp. 239-241], it will be represented as follows:

$$(1.4)\quad H_{p,q}^{m,n}\left[z\left|\begin{matrix}\{a_p,\alpha_p\}\\ \{b_q,\beta_q\}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{T'}\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\zeta)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\zeta)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\zeta)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\zeta)}z^{\zeta}d\zeta,$$

where $\{(f_r,\gamma_r)\}$, stands for the set of the parameters $(f_1,\gamma_1),\cdots,(f_r,\gamma_r)$.

2. In this section, we establish an integral which is required in the development of the present work.

The integral to be established is

$$(2.1)\quad \int_0^1 x^q(1-x)^{-\frac{1}{2}m}e^{-2ux}P_l^{m,n}(2x-1)H_{\gamma,\delta}^{\alpha,\beta}\left[zx^{\mu}\left|\begin{matrix}\{(a_\gamma,\alpha_\gamma)\}\\ \{(b_\delta,\beta_\delta)\}\end{matrix}\right.\right]dx$$

$$=2^{\frac{n-m}{2}}\sum_{N=0}^{\infty}\frac{(-2u)^N}{N!}$$

$$\times H_{\gamma+2,\delta+2}^{\alpha,\beta+2}\left[z\left|\begin{matrix}\left(\frac{n}{2}-q-N,\ \mu\right),\ \left(-\frac{n}{2}-q-N,\ \mu\right),\ \{(a_\gamma,\alpha_\gamma)\}\\ \{(b_\delta,\beta_\delta)\},\ \left(\frac{m}{2}-q+l-N,\ \mu\right),\ \left(\frac{m}{2}-l-q-1-N,\mu\right)\end{matrix}\right.\right],$$

where $\mu>0$, $\mathrm{Re}(m)<1$, $\mathrm{Re}(q+1+\mu b_j/\beta_j)>\frac{1}{2}|\mathrm{Re}\,n|$ $(j=1,2,\cdots,\alpha)$,

$$\sum_1^{\delta}(\beta_j)-\sum_1^{\gamma}(\alpha_j)\geqq 0,\quad \sum_1^{\beta}(\alpha_j)-\sum_{\beta+1}^{\gamma}(\alpha_j)+\sum_1^{\alpha}(\beta_j)-\sum_{\alpha+1}^{\delta}(\beta_j)\equiv\phi>0,\ |\arg\ z|<\frac{1}{2}\phi\pi.$$

PROOF. Expressing the H-function in the integrand as Mellin-Barnes type integral (1.4), interchanging the order in integration, which is justifiable due to the absolute convergence of integrals involved in the process, then evaluating the inner integral with the help of the result [7, p. 288(14)], with x replaced by $(2x-1)$; that is

$$(2.2)\quad \int_0^1(1-x)^{-\frac{m}{2}}x^qe^{-2ux}p_l^{m,n}(2x-1)dx$$

$$=2^{\frac{n-m}{2}}\frac{\Gamma\left(q+\frac{1}{2}n+1\right)\Gamma\left(q-\frac{1}{2}n+1\right)}{\Gamma\left(q-l-\frac{1}{2}m+1\right)\Gamma\left(q+l-\frac{1}{2}m+2\right)}$$

$$\times{}_2F_2\left(q+\frac{1}{2}n+1,\ q-\frac{1}{2}n+1;\ q-l-\frac{1}{2}m+1,\ q+l-\frac{1}{2}m+2;\ -2u\right),$$

where $\mathrm{Re}(m)<1$, $\mathrm{Re}(q+1)>\frac{1}{2}|\mathrm{Re}\ n|$; and applying (1.4), the definition of the H-function, we get the result (2.1).

3. Finite Hankel transforms. Let the finite Hankel transform of $f(r)$ be [8, p. 83]

$$J[f(r)]=\int_0^a rf(r)J_0(r\xi_i)\mathrm{d}r=\bar{f}_j(\xi_i), \tag{3.1}$$

where ξ_i is the root of the transcendental equation

$$J_0(a\xi_i)=0. \tag{3.2}$$

In (2.1), putting $\alpha=1$, $\beta=\gamma=0$, $\delta=2$, $b_1=b_2=\nu$, $\beta_1=\beta_2=1$, $\mu=1$, $x=\frac{r^2}{a^2}$, $z=\frac{\xi_i^2a^2}{4}$ and using [2, p. 434(3) & p. 439(3)], we obtain:

$$J\left[r^{2q+2\nu}(a^2-r^2)^{-\frac{m}{2}}e^{-\frac{2ur^2}{a^2}}P_l^{m,n}\left(\frac{2r^2}{a^2}-1\right)\right] \tag{3.3}$$

$$=2^{\frac{n-m}{2}-1}a^{2q+2\nu-m+2}\sum_{N=0}^{\infty}\frac{\Gamma(1+\nu-\frac{n}{2}+q+N)\Gamma(1+\nu+\frac{n}{2}+q+N)}{\Gamma(1+\nu-\frac{m}{2}+q-l+N)\Gamma(2+\nu-\frac{m}{2}+l+q+N)}$$

$$\times\frac{(-2u)^N}{N!}{}_2F_3\left[\begin{matrix}1+\nu-\frac{n}{2}+q+N,\ 1+\nu+\frac{n}{2}+q+N\\ 1,\ 1+\nu-\frac{m}{2}+q-l+N,\ 2+\nu-\frac{m}{2}+l+q+N\end{matrix};-\frac{\xi_i^2a^2}{4}\right],$$

where $\mathrm{Re}(m)<1$, $\mathrm{Re}(q+1+\nu)>\frac{1}{2}|\mathrm{Re}\ n|$.

(3.3), with $u=0$ gives

$$J\left[r^{2q+2\nu}(a^2-r^2)^{-\frac{m}{2}}P_l^{m,n}\left(\frac{2r^2}{a^2}-1\right)\right] \tag{3.4}$$

$$=\frac{2^{\frac{n-m-2}{2}}a^{2q+2\nu-m+2}\Gamma\left(1+\nu+q-\frac{1}{2}n\right)\Gamma\left(1+\nu+q+\frac{1}{2}n\right)}{\Gamma\left(1+\nu+q-l-\frac{m}{2}\right)\Gamma\left(2+\nu+q+l-\frac{m}{2}\right)}$$

$$\times{}_2F_3\left[\begin{matrix}1+\nu-\frac{n}{2}+q,\ 1+\nu+q+\frac{1}{2}n\\ 1,\ 1+\nu+q-l-\frac{1}{2}m,\ 2+\nu+q+l-\frac{1}{2}m\end{matrix}:-\frac{\xi_i^2a^2}{4}\right],$$

where $\mathrm{Re}(m)<1$ and $\mathrm{Re}(1+q+\nu)>\frac{1}{2}|\mathrm{Re}\ n|$.

By virtue of the inversion theorem [8, p. 83], from (3.3), we get

$$(3.5)\quad r^{2\nu+2q}(a^2-r^2)^{-\frac{m}{2}}e^{-\frac{2ur^2}{a^2}}P_l^{m,n}\left(\frac{2r^2}{a^2}-1\right)$$

$$=2^{\frac{n-m}{2}}a^{2q+2\nu-m}\sum_i\sum_{N=0}^{\infty}\frac{\Gamma\left(1+\nu+q+N+\frac{1}{2}n\right)\Gamma\left(1+\nu+q+N-\frac{1}{2}n\right)}{\Gamma\left(1+\nu+q-l+N-\frac{1}{2}m\right)\Gamma\left(2+\nu+q+l+N-\frac{1}{2}m\right)}$$

$$\times\frac{(-2u)^N}{N!}{}_2F_3\left[\begin{matrix}1+\nu+q+N-\frac{1}{2}n, & 1+\nu+q+N+\frac{1}{2}n\\ 1,\ 1+\nu+q-l+N-\frac{1}{2}m, & 2+\nu+q+l+N-\frac{1}{2}m\end{matrix};-\frac{\xi_i^2a^2}{4}\right]$$

$$\times\frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2},$$

where the sum is taken over all the positive roots of (3.2).

The result (3,5) will prove useful in the verification of the solutions.

4. Solutions of the problem.

We apply finite Hankel transforms (3.3) and (3.4) to obtain solutions of (1.1). By [8, p. 203], solutions are

$$(4.1)\quad u(r,t)=2^{\frac{n-m}{2}}a^{2\nu+2q-m}\frac{k}{K}\sum_i\sum_{N=0}^{\infty}\frac{(-2u)^N}{N!}$$

$$\times\frac{\Gamma\left(1+\nu+q+N+\frac{1}{2}n\right)\Gamma\left(1+\nu+q+N-\frac{1}{2}n\right)}{\Gamma\left(1+\nu+q-l+N-\frac{1}{2}m\right)\Gamma\left(2+\nu+q+l+N-\frac{1}{2}m\right)}$$

$$\times{}_2F_3\left[\begin{matrix}1+\nu+q+N-\frac{1}{2}n, & 1+\nu+q+N+\frac{1}{2}n\\ 1,\ 1+\nu+q-l+N-\frac{1}{2}m, & 2+\nu+q+l+N-\frac{1}{2}m\end{matrix};-\frac{\xi_i^2a^2}{4}\right]$$

$$\times\frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2}h(\xi_i,t),$$

and

$$(4.2)\quad u(r,t)=2^{\frac{n-m}{2}}a^{2\nu+2q-m}\frac{k}{K}\sum_i\frac{\Gamma\left(1+\nu+q+\frac{1}{2}n\right)\Gamma\left(1+\nu+q-\frac{1}{2}n\right)}{\Gamma\left(1+\nu+q-l-\frac{1}{2}m\right)\Gamma\left(2+\nu+q+l-\frac{1}{2}m\right)}$$

$$\times{}_2F_3\left[\begin{matrix}1+\nu+q-\frac{1}{2}n, & 1+\nu+q+\frac{1}{2}n\\ 1,\ 1+\nu+q-l-\frac{1}{2}m, & 2+\nu+q+l-\frac{1}{2}m\end{matrix};-\frac{\xi_i^2a^2}{4}\right]$$

$$\times\frac{J_0(r\,\xi_i)}{[J_1(a\,\xi_i)]^2}h(\xi_i,t),$$

where the sum is taken over all the positive roots of (3.2); and

$$h(\xi_i,t)=\int_0^t g(T)\exp\{-k\xi_i^2(t-T)\}dT\ . \tag{4.3}$$

5. Verification of the solution.

With the help of (4.1) and [5, p. 100(5.2.4) & (5.2.5)], we get the value of $\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$; using (1.2) and (3.5) we have the value of $\theta(r,t)$ and from (4.1) we can obtain the value of $\frac{\partial u}{\partial t}$. On substituting these values in (1.1), we see that the equation is satisfied.

The boundary condition $u(a,t)=0$ is satisfied, as $J_0(a\ \xi_i)$ is zero and it is present in every term of $u(a,t)$. The initial condition is satisfied as $h(\xi_i,0)=0$.

As (4.1) converges uniformly when $t>0$, so the function $u(r,t)$ given by its is continuous when $0\leq r\leq a$.

The term by term differentiations are justified as the values of $\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$ and $\frac{\partial u}{\partial t}$ are uniformly convergent for $t>0$ and $0\leq r<a$.

6. Heat source of general character.

Let

$$g(T)=g_0T^q(t-T)^{-\frac{m}{2}}e^{-Tz}P_l^{m,n}\left(\frac{2T}{t}-1\right)$$
$$\times H_{\gamma,\delta}^{\alpha,\beta}\left[\lambda T^{\mu}\middle|\begin{matrix}\{(a_\gamma,\alpha_\gamma)\}\\\{(b_\delta,\beta_\delta)\}\end{matrix}\right]. \tag{6.1}$$

Substituting the value of g(T) in (4.3) and using (2.1), we get

$$h(\xi_i,t)=2^{\frac{n-m}{2}}g_0\,t^{1+q-\frac{m}{2}}e^{-k\xi_i^2t}\sum_{h=0}^{\infty}\frac{(k\,\xi_i^2-z)^h\,t^h}{h!} \tag{6.2}$$
$$\times H_{\gamma+2,\delta+2}^{\alpha,\beta+2}\left[\lambda t^{-\mu}\middle|\begin{matrix}\left(\frac{n}{2}-q-h,\mu\right),\ \left(-\frac{n}{2}-q-h,\mu\right),\ \{(a_\gamma,\alpha_\gamma)\}\\\{(b_\delta,\beta_\delta)\},\ \left(\frac{m}{2}-q+l-h,\mu\right),\ \left(\frac{m}{2}-q-l-1-h,\mu\right)\end{matrix}\right],$$

where $\mathrm{Re}(m)<1$, $\mathrm{Re}(q+1+\mu b_j/\beta_j)>\frac{1}{2}|\mathrm{Re}\ n|$ $(j=1,2,\cdots,\alpha)$, $\mu>0$,

$\sum_1^{\delta}(\beta_j)-\sum_1^{\gamma}(\alpha_j)\geq 0$, $\sum_1^{\beta}(\alpha_j)-\sum_{\beta+1}^{\gamma}(\alpha_j)+\sum_1^{\alpha}(\beta_j)+\sum_1^{\alpha}(\beta_j)-\sum_{l+1}^{\delta}(\beta_j)\equiv\phi>0$, $|\arg\lambda|<\frac{1}{2}\phi\pi$. Using (6.2) in (4.1), we obtain

$$(6.3)\qquad u(r,t)=2^{n-m}a^{2\nu+2q-m}\,t^{1+q-\frac{m}{2}}g_0\frac{k}{K}\sum_i\sum_{N=0}^{\infty}\sum_{h=0}^{\infty}\frac{(-2u)^N}{N!}$$

$$\times\frac{\Gamma\left(1+\nu+q+N+\frac{1}{2}n\right)\Gamma\left(1+\nu+q+N-\frac{1}{2}n\right)}{\Gamma\left(1+\nu+q-l+N-\frac{1}{2}m\right)\Gamma\left(2+\nu+q+l+N-\frac{1}{2}m\right)}\cdot e^{-k\xi_i^2 t}$$

$$\times{}_2F_3\left[\begin{matrix}1+\nu+q+N-\frac{1}{2}n,\ 1+\nu+q+N+\frac{1}{2}n\\ 1,\ 1+\nu+q-l+N-\frac{1}{2}m,\ 2+\nu+q+l+N-\frac{1}{2}m\end{matrix}:-\frac{\xi_i^2a^2}{4}\right]$$

$$\times H_{\gamma+2,\delta+2}^{\alpha,\beta+2}\left[\frac{\lambda}{t^{\mu}}\left|\begin{matrix}\left(\frac{n}{2}-q-h,\mu\right),\ \left(-\frac{n}{2}-q-h,\mu\right),\ \{(a_\gamma,\alpha_\gamma)\}\\ \{(b_\delta,\beta_\delta)\},\ \left(\frac{m}{2}-q+l-h,\mu\right),\ \left(\frac{m}{2}-q-l-1-h,\mu\right)\end{matrix}\right.\right]$$

$$\times\frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2}\cdot\frac{(k\xi_i^2-z)^h t^h}{h!}.$$

Obviously $u(r,0)=0$.

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