ON THE COMMUTATIVITY OF NEAR RINGS

By Steve Ligh

A near ring is a triple $(R,+,\cdot)$ such that $(R,+)$ is a group, $(R,\cdot)$ is a semigroup and $\cdot$ is left distributive over $+$, i.e. $w(x+y)=wx+wy$ for each w, x, $y\in R$. An element $r\in R$ is called right distributive if and only if $(a+b)r=ar+br$ for each a, $b\in R$. A near ring R is distributively generated (d.g.) if there exists $S\subset R$ such that $(S,\cdot)$ is a sub-semigroup of $(R,\cdot)$, each element of S is right distributive, and S is an additive generating set for $(R,+)$. Other definitions may be found in [4]. The purpose of this note is to extend the famous "$x^{n(x)}=x$" theorem of Jacobson to a special class of d.g. near rings. Also a very elementary proof of theorem 2 in [3] is obtained.

DEFINITION. A d.g. near ring R is called an *α-near ring* if whenever $x\in R$ is right distributive, then $-x$ is also right distributive.

Every ring is clearly an α-near ring. Examples of α-near rings which are not rings can be found in [1, 2.5, #29, #36].

The following lemma is easy.

LEMMA 1. *If R is a near ring and x is a right distributive element, then $(-w)x=-(wx)=w(-x)$ for each $w\in R$.*

LEMMA 2. *Let R be an α-near ring and suppose x is a right distributive element in R. Then $yx=0$ for each $y\in R'$, where R' is the commutator subgroup of $(R,+)$.*

PROOF. It suffices to show that $(a+b-a-b)x=0$ for each a, $b\in R$. This follows from the calculation below.

$$\begin{aligned}(a+b-a-b)x&=[a-(a-b)-b]x=ax+[-(a-b)]x+(-b)x\\&=ax+(a-b)(-x)+(-b)x\\&=ax+a(-x)+(-b)(-x)+(-b)x=0.\end{aligned}$$

THEOREM. *Let R be an α-near ring without any nonzero nilpotent elements. Then R is a ring.*

PROOF. Since R is d.g., every element in R can be written as a finite sum of right and anti-right distributive elements [4, p. 1367]. In particular, R is an α-near ring, so each element of R is a finite sum of right distributive elements.

Now suppose $x \neq 0$ is an element of R' and $x = a_1 + a_2 + \cdots + a_n$. Then by lemma 2, we have that

$$x^2 = x(a_1 + a_2 + \cdots + a_n) = xa_1 + xa_2 + \cdots + xa_n = 0.$$

This contradiction implies that $R' = 0$ and hence $(R, +)$ is an abelian group. It follows that [2, p. 93] R is a ring.

Using the "$x^{n(x)} = x$" theorem of Jacobson we get the following result.

COROLLARY 1. *Let R be an α-near ring such that for each $x \in R$, there is an integer $n(x) > 1$ for which $x^{n(x)} = x$. Then R is a commutative ring.*

The next result [3, Theorem 2] was obtained by using subdirect sum representation of near rings. The following is a very elementary proof.

COROLLARY 2. *Every d. g. boolean near ring R is a boolean ring.*

PROOF. Suppose that x is a right distributive element. Since $(x+x)^2 = x+x$, we see that $x+x=0$ by expanding $(x+x)^2$. Thus $-x = x$ and hence $-x$ is also right distributive. It follows that R is an α-near ring and by corollary 1, R is a boolean ring.

A near ring R is said to be distributive if every element of R is right distributive. Thus R is an α-near ring. An example of a distributive near ring which is not a ring is given in [1, 2.5, #29]. The following is a corollary of lemma 2.

COROLLARY 3. *Let R be a distributive near ring. Then either every element of R is a zero divisor or R is a ring.*

University of Florida

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