QUASIREGULARITIES AND QUASIREGULAR RELATIONS

By D. V. Thampuran

It is well known (see Császár [1], Smirnov [2]) that the topological space of a proximity relation is completely regular and that every proximity is generated by a uniformity; quasiproximities have similar properties. The object of this paper is to prove the analogs of these properties for quasiregular relations.

Let X be a set. For a subset A of X write $cA=X-A$. If $b\subset 2^X\times 2^X$ then $(A,B)\in b$ will also be denoted by $(A,B)\in cb$; if A,B consist of single points x,y we will write x,y for A,B.

DEFINITION 1. $p\subset 2^X\times 2^X$ is said to be a *quasineighborhood relation* for X and (X,p) a *quasineighborhood space* iff for all x in X and all subsets A,B,C,D of X

1. $(X,\phi),\quad(\phi,X)\in cp$
2. $(A,B)\in p, A\subset C, B\subset D$ imply $(C,D)\in p$ and
3. $A\cap B\neq\phi$ implies $(A,B)\ \varepsilon\ p$.

A quasineighborhood relation q is said to be a *quasiregular* relation iff

4. $(A\cup B,C)\in q$ implies (A,C) or $(B,C)\in q$; $(A,B\cup C)\in q$ implies (A,B) or $(A,C)\in q$ and
5. $(x,B)\in cq$ implies $(x,S),\ (cS,B)\in cq$ for some $S\subset X$; $(A,x)\in cq$ implies $(A,cT),\ (T,x)\in cq$ for some $T\subset X$.

Let q be a quasiregular relation for X. For a subset A of X define $kA=\{x:(A,x)\in q, x\in X\}$ and $k'A=\{x:(x,A)\in q, x\in X\}$. Obviously k,k' are Kuratowski closure functions for X.

DEFINITION 2. With k,k' as defined above, (X,q,k,k') is said to be the *bitopological space* of q.

It is easily seen that the bitopological space of a quasiregular relation is regular.

Let A be a subset of X and $U\subset X\times X$, Write $AU=\{y:(x,y)\in U$ for some $x\in A\}$ and $UA=\{y:(y,x)\in U$ for some $x\in A\}$; if A contains only one point x we will write xU,Ux for $AU,\ UA$. We will also write xUU for $(xU)U$ and UUx for $U(Ux)$.

Let $\mathscr{U}$ be a family of subsets of $X \times X$ such that for x in X

1. (x, x) is in each member of $\mathscr{U}$
2. U in $\mathscr{U}$ implies there is V in $\mathscr{U}$ such that $xVV \supset xU$and $VVx \subset Ux$
3. U, V in $\mathscr{U}$ imply $U \cap V$ is in $\mathscr{U}$
4. $U \subset V \subset X \times X$ and U in $\mathscr{U}$ imply V is in $\mathscr{U}$.

DEFINITION 3. $\mathscr{U}$, as defined above, is said to be a *quasiregularity* for X. For subsets A, B of X take $(A, B) \in q$ iff $AU \cap B \neq \phi$ for each U in $\mathscr{U}$. Then q which is quasiregular, is said to be the *quasiregular relation* of $\mathscr{U}$.

DEFINITION 4. Let (X, p), (Y, s) be two quasineighborhood spaces and f a function from X to Y. Then f is said to be an *n-function* iff $(A, B) \in p$ implies $(fA, fB) \in s$.

If f is an n-function from a bineighborhood space (X, p, g, g') to a bineighborhood space (Y, s, h, h') then it is obvious that f is continuous.

Let $N = \{1, 1/2, 1/3, \cdots\cdots, 0\}$. Define a distance function e for N as follows. For all u, v in N

$$e(u, v) = \begin{cases} v-u & \text{if } u<w<v \text{ for some } w \text{ in } N \\ 0 & \text{otherwise.} \end{cases}$$

For subsets A, B of N take $e(A, B) = \inf\{e(u, v)\colon u \in A,\ v \in B\}$. Define the *quasineighborhood relation* n for N as follows: $(A, B) \in n$ iff $e(A, B) = 0$. Let $(A, B) \in n'$ iff $(B, A) \in n$.

Thampuran [3] has proved the following result. Let q be a quasiregular relation for X. Then

1. $(x, B) \in cq$ implies there is an n-function from (X, q) to (N, n) such that $f(x)$ is 0 and f is 1 on B and
2. $(A, x) \in cq$ implies there is an n-function from (X, q) to (N, n') such that $f(x)$ is 0 and f is 1 on A.

THEOREM. *Let q be a quasiregular relation for X. Then there is a quasiregularity $\mathscr{U}$ for X such that the quasiregular relation p of $\mathscr{U}$ has the properties:*

1. $q \subset p$
2. (A, x), $(x, B) \in q$ *iff* (A, x), $(x, B) \in p$ *and*
3. p *and* q *have the same bitopological space.*

PROOF. For each $(x, B) \in cq$ there is an n-function f from (X, q) to (N, n) such that $f(x)$ is 0 and f is 1 on B and so there is a d for X defined by $d(x', y') = e(f(x'), f(y'))$ for all x', y' in X; let D be the family of all such d. For each $(A, x) \in cq$ there is an n-function f' from (X, q) to (N, n') such that $f(x)$ is 0 and f is 1 on A and so there is a d' for X defined by $d'(x', y')=e(f(y'), f(x'))$ for all x', y' in X; let D' be the family of all such d'. Take $E=D\cup D'$.

For d in D and $r>0$ take $V(d, r)=\{(x, y): d(x, y)<r, x, y \text{ in } X\}$. Consider a $U=V(d, r)$ and let x be a point of X. Let $B=c(xU)$. It is easily seen $(x, B)\in cq$ and so (x, S), $(cS, B) \in cq$ for some $S\subset X$. Hence there is an n-function f from (X, q) to (N, n) such that $f(x)$ is 0 and f is 1 on S. For y, z in X take $a(y, z)=e(f(y), f(z))$. Let $V=V(a, 1/8))$ and t in xV. Then $a(x, t)<1/8$ and so $f(t)<1/8$. Also $u \in tV$ implies $f(u)<1/4$ and so $u \varepsilon cS\subset cB$. Hence $xVV\subset xU$. Similarly we can prove that $V'V'x\subset Ux$ for some $V'(a', 1/8)$, $a' \varepsilon D'$.

Let $\mathscr{U}$ be the family of all subsets U of $X\times X$ such that U contains the intersection of a finite number of the sets $V(d, r)$ for d in E and $r>0$. It is obvious that $\mathscr{U}$ is a quasiregularity for X.

Let $(A, B) \in q$ and take a $U=V(d, r)$ for d in E and $r>0$, Since d is obtained from an n-function it is obvious that $AU\cap B\neq\phi$ and so $(A, B) \in p$. Next, $(x, B) \in cq$ implies there is a d in E such that $U=V(d, 1/4)$ implies $xU\cap B=\phi$ and so $(x, B) \in cp$. Similarly$(A, x) \in cq$ implies $(A, x) \in cp$. Hence p and q have the same bitopological space.

State University of New York
at Stony Brook

REFERENCES

[1] Á. Császár, *Foundations of general topology*, New York (1963).

[2] Yu. M. Smirnov, *On proximity spaces*, Mat. Sb. 31 (73) (1952) 543—574 (Russian).

[3] D. V. Thampuran, *Quasiregular relations and functional separation* (to appear).