A NOTE ON ATOMIC MEASURE AND SINGULARITY

By Jung-Soo Woo and In-Suk Lee

In papers [1] and [3], R. A. Johnson left open questions that;

(1) If $\{\mu_n\}$ is a sequence of atomic measures, are $\mu=\sum\mu_n$ and $\vee\mu_n$ atomic?

(2) Isn't the regularity of Borel measure μ or ν essential in the following (I) and (II)?

(I) (1) If $\mu_\alpha(E)\uparrow\mu_\alpha(E)$ for each Borel set E, (2) $\mu_\alpha\perp\nu$ for each α and (3) μ or ν is regular, then $\mu\perp\nu$. Hence, if (1) $\mu=\sum\mu_\alpha$ or $\mu=\vee\mu_\alpha$, (2) $\mu_\alpha\perp\nu$ for each α and (3) μ or ν is regular, then $\mu\perp\nu$.

(II) Lebesgue decomposition theorem: Suppose μ and ν are Borel measures on a locally compact Hausdroff space X such that μ or ν is regular. Then there exist unique Borel measures μ_0 and μ_1 such that $\mu=\mu_0+\mu_1$, where $\mu_0\ll\nu$ and $\mu_1\perp\nu$.

The purpose of this paper is to give positive answers for above questions, and to obtain that ν is S-singular with respect to μ if and only if ν is singular with respect to μ under the condition that ν is σ-finite measure. Next we get an extension of the form of the Lebesgue extension theorem given by R. A. Johnson [2].

DEFINITIONS. A set $A\subset X$ is *locally measurable* if $A\cap E$ is measurable for each measurable set ([5], p. 35).

A set $E\in S$ will be called an *atom* for μ if (1) $\mu(E)>0$ and (2) given $F\in S$, either $\mu(E\cap F)$ or $\mu(E-F)$ is 0.

μ is called to be *purely atomic* or *simple atomic* if every measurable set of positive measure contains an atom, and μ is called to be non-atomic if there are no atoms for μ.

We shall say that ν is *S-singular* with respect to μ, denoted $\nu S\mu$, if given $E\in S$ there exists a measurable set $F\subset E$ such that $\nu(E)=\nu(F)$ and $\mu(F)=0$.

Other terminology and definitions in this note are consistent with those used in Halmos [4].

LEMMA 1. *If* $\{\nu_n\}$ *is a increasing sequence of atomic measures and* $\lim\nu_n=\nu$, *then ν is atomic.*

PROOF. Suppose ν is not atomic. If $\nu(E)>0$, there exists a set $G\in S$ such that $\nu(E\cap G)>0$ and $\nu(E-G)>0$. Hence there exists integer "m" such that ν_n is not atom for each $n\geqq m$. This is a contradiction.

The following theorem gives positive answer of question (1).

THEOREM 2. *If $\{\mu_n\}$ is a sequence of atomic measures, then $\sum\mu_n$ and $\vee\ \mu_n$ are atomic.*

PROOF. Let $\nu_n=\sum_{i=1}^{n}\mu_i$ ($\vee\{\mu_i;\ i=1,2,\cdots\cdots n\}$). Then ν_n is atomic for each n from the result of R. A. Johnson [1, Theorem 1.1]. Thus $\sum\mu_n$ and $\vee\ \mu_n$ are atomic.

THEOREM 3. *Let μ be a measure on a σ-ring S, ν be σ-finite measure on S. Then ν is singular with respect to μ if and only if ν is S-singular with respect to μ.*

PROOF. Suppose that ν is S-singular with respect to μ. For given measurable set E, there exists a disjoint sequence $\{E_i\}$ such that $E=\bigcup E_i$ and $\nu(E_i)<\infty$ for each i. For each i, there exists a measurable set $F_i\subset E_i$ such that $\nu(F_i)=\nu(E_i)$ and $\mu(F_i)=0$. Let $F=\bigcup F_i$, $\mu(E\cap F)=0$ and $\nu(E-F)=\nu(\bigcup E_i-\bigcup F_i)\leqq\nu(\bigcup_i(E_i-F_i))=\sum\nu(E_i-F_i)=\sum(\nu(E_i)-\nu(F_i))=0$.
Thus ν is singular with respect to μ.

Conversely if ν is singular with respect to μ, then it is clear that ν is S-singular with respect to μ.

REMARK. R. A. Johnson ([2], p. 631) gave a counter example such that $\nu S\mu$ does not imply $\nu\perp\mu$, even for σ-finite measure. But above theorem 3 shows that the example is false and we can easily prove that ν is singular with respect to μ at that example.

From the result of R. A. Johnson ([2], theorem 3.1) and theorem 3, we have the following.

THEOREM 4. *If $\{\mu_\alpha\}$ is a increasing directly family of Borel measures on locally compact Hausdroff space X such that $\mu_\alpha\uparrow\mu$ and $\mu_\alpha\perp\nu$ for each α and fixed Borel measure ν, then μ is singular with respect to ν.*

Above theorem4 gives positive answer of question ((2), I)

From theorem 4 and the fact that finite sum of μ_α is singular with respect to ν.

Hence we have

COROLLARY 5. *Let* $\{\mu_\alpha\}$ *is a family of Borel measures. If* $\mu_\alpha \perp \nu$ *for each* α *and* $\mu = \sum \mu_\alpha$, *then* μ *is singular with respect to* ν.

Next we show an extension of the form of the Lebesgue decomposition theorem given by R. A. Johnson ([2], theorem 3.4)

THEOREM 6. *If* μ *is a measure on a* σ*-ring* S *and* ν *is* σ*-finite measure on* S, *then there exist unique measures* ν_0 *and* ν_1 *such that* $\nu = \nu_0 + \nu_1$, *where* $\nu_0 \ll \mu$, $\nu_1 \perp \mu$ *and* $\nu_0 \perp \nu_1$.

PROOF. Let $\mathfrak{M} = \{M \in S\colon \mu(M) = 0\}$. Clealy $\mathfrak{M}$ is closed under countable unions. Define a measure ν_1 on S by $\nu_1(E) = LUB\{\nu(E \cap M); M \in \mathfrak{M}\}$.

Then there exists $\widetilde{M} \in \mathfrak{M}$ such that $\widetilde{M} \subset E$, $\nu_1(E) = \nu_1(\widetilde{M})$ and $\mu(\widetilde{M}) = 0$. Since ν is σ-finite and $\nu_1 \leqq \nu$, there exists a disjoint sequence $\{E_n\}$ such that $E = \bigcup E_n$ and $\nu_1(E_n) < \infty$ for each n, thus there exists $M_n \in \mathfrak{M}$ such that $M_n \subset E_n$, $\nu_1(E_n) = \nu_1(M_n)$ and $\mu(M_n) = 0$ for each n. Let $M = \bigcup M_n$, we obtain $\mu(E \cap M) = 0 = \nu_1(E - M)$. Thus $\nu_1 \perp \mu$

Let $\mathfrak{N} = \{N \in S;\ \nu_1(N) = 0\}$, obviously $\mathfrak{N}$ is closed under countable unions. Define a measure ν_0 on S by $\nu_0(E) = LUB\{\nu(E \cap N);\ N \in \mathfrak{N}\}$

Then $\nu_0 \ll \mu$ and $\nu = \nu_0 + \nu_1$. Now suppose $\nu_0 + \nu_1$ and $\nu_2 + \nu_3$ are two decompositions of ν. Since $\nu_2 \ll \mu$ and $\nu_1 \perp \mu$, we get $\nu_1 \perp \nu_2$. Similarly ν_0 is singular with respect to ν_3. Since $\nu_1 \leqq \nu_2 + \nu_3$ and $\nu_1 \perp \nu_2$, we have $\nu_1 \leqq \nu_3$. $\nu_3 \leqq \nu_1$ follows by similar construction. Therefore we get $\nu_1 = \nu_3$. Similarly we obtain $\nu_0 = \nu_2$, which completes proof.

Thus we have

COROLLARY 7. If μ and ν are Borel measures on locally compact Hausdroff space X, then we have the same decomposition.

Above corollary 7 gives positive answer of question ((2), II).

Kyungpook University

REFERENCES

[1] R.A. Johnson, *An atomic and non-atomic measures*, Proc. Amer. Math. Soc. (1970), 650—655.

[2] R.A. Johnson, *On the Lebesgue decomposition theorem*, Proc. Amer. Math. Soc. (1967), 628—632.

[3] R.A. Johnson, *Some types of Borel measures*, Proc. Amer. Math. Soc. (1969), 94—99.

[4] P.R. Halmos, *Measure theory*, Van. Nostrand, New York, (1950).

[5] S.K. Berberian, *Measure and Integration*, Macmillan, New York, (1965).