NOTES ON ATOMIC MEASURES

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In this paper, we give a simple characterization for the atomic measure, and using this characterization and results of Johnson [1], we shall show that the product measure of two atomic measures is also atomic.

Our definition for atomic and nonatomic measures coincides with those of Johnson [1]. Other definition and terminology follow those in Halmos [2].

DEFINITIONS. A subset A of measurable space $(X, \mathscr{S})$ is called a *locally measurable set* if, for each $E \in \mathscr{S}$, $E \cup A$ is a measurable set.

Suppose μ is a measure on the σ- ring $\mathscr{S}$. A set E will be called an *atom* for μ if (1) $\mu(E) > 0$ and (2) given $F \in \mathscr{S}$, either $\mu(E \cap F)$ or $\mu(E - F)$ is 0.

In order to prove the equivalent condition for the atomic measure, we have the following.

LEMMA 1. *If E is a measurable set of positive measure in measure space $(X, \mathscr{S}, \mu)$ and $\mathfrak{M}(E)$ is the class of all locally measurable sets A for which either $\mu(E \cap A)$ or $\mu(E - A)$ is 0, then $\mathfrak{M}(E)$ is a monotone ring.*

PROOF. To show that $\mathfrak{M}(E)$ is a ring, we assume that A and B are arbitrary members of $\mathfrak{M}(E)$. Then, by the relations $E - (A - B) = (E - A) \cup (E \cap B)$ and $E \cap (A - B) = (E - B) \cap (E \cap A)$, and the facts that either $\mu(E - A)$ or $\mu(E \cap A)$ is 0 and either $\mu(E - B)$ or $\mu(E \cap B)$ is 0, we get that either $\mu(E - (A - B))$ or $\mu(E \cap (A - B))$ is 0. It implies that $A - B \in \mathfrak{M}(E)$.

The fact that either $\mu(E - (A \cup B))$ or $\mu(E \cap (A \cup B))$ is 0 follows from the relation that $E - (A \cup B) = (E - A) \cap (E - B)$ and $E \cap (A \cup B) = (E \cap A) \cup (E \cap B)$, and the fact that A and B are members of $\mathfrak{M}(E)$. Hence we get $A \cup B \in \mathfrak{M}(E)$.

Now we show that $\mathfrak{M}(E)$ is a monotone class. Let $\{A_n\}$ be an incresing sequence in $\mathfrak{M}(E)$. If there is a set A_n such that $\mu(E - A_n) = 0$, then $\mu(E - (\cup_n A_n)) = 0$. On the other hand, if, for each member A_n in $\{A_n\}$, we have $\mu(E \cap A_n) = 0$, then $\mu(E \cap (\bigcup_n A_n)) = \mu(\lim_n E \cap A_n) = \lim_n \mu(E \cap A_n) = 0$.

Next suppose $\{B_n\}$ be a decreasing sequence in $\mathfrak{M}(E)$. If there exists a member B_n such that $\mu(E\cap B_n)=0$, then $\mu(E\cap(\bigcap_n B_n))=0$. If, for each n, we have $\mu(E-B_n)= =0$, then $\mu(E-\bigcap_n B_n)=\mu(\lim_n E-B_n)=\lim_n \mu(E-B_n)=0$. These facts imply that $\mathfrak{M}(E)$ is a monotone class.

From Lemma 1 and the result that a monotone ring is σ-ring [2, 6. A], we obtain easily the following.

THEOREM 2. *Suppose* $(X, \mathscr{S}, \mu)$ *is a measure space, and* σ*-ring* $\mathscr{S}$ *is generated by the class of sets* $\mathscr{F}$. *Then the following are equivalent.*
(i) E *is an atom for measure* μ.
(ii) *for each member* F *of* $\mathscr{F}$, *either* $\mu(E\cap F)$ *or* $\mu(E-F)$ *is* 0.

Now, we shall show that the product measure of two atomic measures is atomic by the aid of above result. In order to show this, we prove following two lemmas.

DEFINITION. We shall say that $(X, \mathscr{S}, \mu)$ is an *atomic measure space* if every measurable set of positive measure contains an atom.

LEMMA 3. *Suppose* $(X, \mathscr{S}, \mu)$ *and* $(Y, \mathscr{T}, \nu)$ *are* σ*-finite atomic measure spaces and, let* E *and* F *be atoms for* μ *and* ν *respectively. Then measurable rectangle* $E\times F$ *is an atom for product measure* $\mu\times\nu$.

PROOF. Let $R\times S$ be a measurable rectangle. Since $E\times F-R\times S=[(E\cap R)\times(F-S)]\cup[(E-R)\times F]$ and $(E\times F)\cap(R\times S)=(E\cap R)\times(F\cap S)$. The facts that either $\mu(E\cap R)$ or $\mu(E-R)$ is 0, and either $\nu(F\cap S)$ or $\nu(F-S)$ is 0, imply that either $\mu\times\nu((E\times F)-(R\times S))$ or $\mu\times\nu((E\times F)\cap(R\times S))$ is 0.

Since $\mathscr{S}\times\mathscr{T}$ is σ-ring generated by the class of all measurable rectangles. By Theorem 2, we get $E\times F$ is $\mu\times\nu$-atom.

LEMMA 4. *Suppose* $(X, \mathscr{S}, \mu)$ *and* $(Y, \mathscr{T}, \nu)$ are σ*-finite atomic measure spaces. Then, for each measurable rectangle* $E\times F$ *with positive measure, there exist countable disjoint collection of atoms* $E_k\times F_k$ *for* $\mu\times\nu$ *such that* $\mu\times\nu(E\times F)=\mu\times\nu(\bigcup_k E_k\times F_k)$ *and* $E_k\times F_k\subset E\times F$ *for each* k.

PROOF. It is sufficient to show the case that $\mu(E)<\infty$ and $\nu(F)<\infty$. Now since $\mu(E)>0$, $\nu(F)>0$, $E\in\mathscr{S}$ and $F\in\mathscr{T}$, there exists a countable collection of atoms $E_m\subset E$ ($F_n\subset F$) such that $\mu(E)=\mu(\bigcup_m E_m)$ ($\nu(F)=\nu(\bigcup_n F_n)$ respectively) [1, Theorem 2. 2].

Since $\mu\times\nu(E\times F)=\mu(\bigcup_m E_m)\times\nu(\bigcup_n F_n)=\mu\times\nu(\bigcup_{m,n} E_m\times F_n)$ and each $E_m\times F_n$ is an atom for $\mu\times\nu$ by lemma 3, the countable collection $\{E_m\times F_n\}$ is required one.

THEOREM 5. *If* $(X\times Y,\ \mathscr{S}\times\mathscr{T},\ \mu\times\nu)$ *is a product measure space of two σ-finite atomic measure spaces* $(X,\mathscr{S},\mu)$ *and* $(Y,\mathscr{T},\nu)$, *then* $(X\times Y,\ \mathscr{S}\times\mathscr{T},\ \mu\times\nu)$ *is also σ-finite atomic.*

PROOF. If E is arbitrary member of $\mathscr{S}\times\mathscr{T}$ with positive measure, then there exists a sequence $\{E_n\}$ of measurable rectangle of finite measure whose unon contains measurable set E. By lemma 4, there exists a countable disjoint collection of atoms E_n^k such that $\mu\times\nu(E_n)=\mu\times\nu(\bigcup_k E_n^k)$ for each n. Hence, there is at least one E_n^k such that $\mu\times\nu(E_n^k\cap E)>0$; otherwise $\mu\times\nu(E)=0$. Now since any subset of atom of positive measure is also an atom [1, p. 650], $E_n^k\cap E$ is an atom for $\mu\times\nu$. It follows that every member of $\mathscr{S}\times\mathscr{T}$ contains an $\mu\times\nu$-atom.

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