MUTANTS IN SEMIGROUPS, A GENERALIZATION

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1. Introduction.

A mutant in a semigroup is defined as follows: a subset M of a semigroup S is a ***mutant*** if and only if $MM \subseteq S \backslash M$, where $MM = \{ab: a \in M \text{ and } b \in M\}$ and $S \backslash M$ is the set of all elements in S not in M. In [2] Iseki made a definition of a mutant in a semigroup S as follows: a subset M of S is an (m, n) mutant of S if and only if $M^m \subset S \backslash M^n$. Iseki also established in [2] a theorem which states that if H an L are (m, n) mutants in semigroups S and T, respectively, then $H \times L$ is an (m, n) mutant of $S \times T$. The main purpose of this paper is to generalize the concept of mutant by defining a set we choose to call a ***niltant***, and proving a theorem similar to Theorem 2 in [3] which Kim has established.

2. Notations and Definitions.

Let S be a semigroup. If $H \subseteq S$, define $E(H) = \{e \in H: e^2 = ee = e\}$. For n a positive integer let $H_n = \{a_1 a_2 \cdots\cdots a_n: a_i \in H, 1 \leq i \leq n\}$ and $S \backslash H = \{a \in S: a \notin H\}$.

DEFINITION. Let N be a subset of a semigroup S. If $N^i \subseteq S \backslash N$ for all positive integers i, $1 < i \leq n+1$ (n is a fixed integer), we then say N is a *n-th order niltant* in S.

NOTE. A 1-st order niltant is a mutant.

3. Theorems.

We will now state and prove a theorem similar to the theorem established by Iseki which was stated in our introduction. But first we will need a lemma.

LEMMA. *If M and N are n-th order niltants in semigroups S and T, respectively, then $M \times N$ is a n-th order niltant in $S \times T$.*

PROOF. Let $(x_i, y_i) \in M \times N$, $i = 1, 2, \cdots\cdots r$, for $1 < r \leq n+1$. Now suppose $(x_1, y_1)(x_2, y_2) \cdots\cdots (x_r, y_r) = (x_1 x_2 \cdots\cdots x_r, y_1 y_2 \cdots\cdots y_r) \in M \times N$.

This implies $(x_1 x_2 \cdots\cdots x_r) \in M$ and $(y_1 y_2 \cdots\cdots y_r) \in N$ which gives a contradiction. Hence our conclusion holds.

THEOREM 1. *Let M be a m-th order niltant, N a n-th order niltant, in semigroups S and T respectively. Then* $M\times N$ *is a p-th order niltant in* $S\times T$, *where p is the max of m and n.*

PROOF. If $m=n$, our conclusion follows from the lemma. Suppose then that $m<n$. Let $(x_i, y_i)\in M\times N$ for $i=1, 2, \cdots\cdots r$ where $1<r\leq n+1$. Now if

$$(x_1, y_1)(x_2, y_2)\cdots\cdots(x_r, y_r)\in M\times N,$$

then $(y_1 y_2\cdots\cdots y_r)\in N$. But $N^r\subseteq S\backslash N$, hence a contradiction. Dually we get a contradiction for $m>n$. This completes our proof.

Before we state the main result of this paper, we reproduce here Theorem 2 of Kim in [3], without the proof.

THEOREM. *Let S be a semigroup.*

(i) *S has no decomposition* $S=M_1\cup M_2$,

(ii) *S has no decomposition* $S=M_1\cup M_2\cup M_3$ *into three disjoint mutants* M_i, $(i=1, 2, 3)$ *of S.*

We shall now state and prove the main theorem of this paper.

THEOREM 2. *A semigroup S has no decomposition into the union of three disjoint n-th order niltants in S, for* $n\geq 1$.

PROOF. Clearly, if $E(S)\neq\phi$, the empty set, then our conclusion follows quickly. Thus we will assume $E(S)=\phi$, which also implies S is infinite. Now the combination of our note at the end of section 2 and Kim's Theorem 2 stated above, gives our desired result for $n=1$.

Consider now N_1, N_2, N_3 three disjoint 2-nd order niltants in S. The following symbol borrowed from Kim [3] will be used:

(1, 4,)
(2, 5,)
(3,) (6)

This symbol denotes that if niltant N_1 contains elements x, x^4, niltant N_2 contains x^2, x^5, and niltant N_3 contains x^3. Then there is no niltant $N_i (i=1, 2, 3)$ containing x^6.

We have the following combinations for $n=2$.

(1, 4,)	(1, 4,)	(1, 5,)	(1, 6,)	(1, 6, 9,)
(2, 5,)	(2,)	(2,)	(2, 5,)	(2, 7, 8,)

(3,)(6) (3, 5,)(6) (3, 4,)(6) (3, 4,)(7) (3, 4, 5,)(10)

(1, 4, 7, 10, 13,) (1, 4, 7,)
(2, 3, 11, 12,) (2, 3, 10, 11,)
(5, 6, 8, 9,)(14) (5, 6, 8, 9,)(12)

(1, 4, 10, 13,) (1, 4, 11,)
(2, 3, 11, 12,) (2, 3, 10,)
(5, 6, 7, 8, 9,)(14) (5, 6, 7, 8, 9,)(12)

(1, 4,) (1, 5, 8, 12,)
(2, 3, 10, 11,) (2, 3, 10, 11,)
(5, 6, 7, 8, 9,)(12) (4, 6, 7, 9,)(13)

(1, 6,) (1, 7,) (1, 8,)
(2, 3,) (2, 3,) (2, 3,)
(4, 5, 7,)(8) (4, 5, 6,)(8) (4, 5, 6, 7,)(9)

Our conclusion for $n=2$.

Consider $n=3$.

(1, 5,) (1, 6,) (1, 6,) (1, 5,)
(2,) (2, 5,) (2, 7,) (2, 3,)
(3, 4,)(6) (3, 4,)(7) (3, 4, 5,)(8) (4, 6, 7,)(8)

(1, 6,) (1, 7,) (1, 8,)
(2, 3,) (2, 3,) (2, 3,)
(4, 5, 7,)(8) (4, 5, 6,)(8) (4, 5, 6, 7,)(9)

Our conclusion for $n=3$.

Consider $n=4$.

(1, 6,) (1, 6,) (1, 6,)
(2, 5,) (2, 7,) (2, 3,)
(3, 4,)(7) (3, 4, 5,)(8) (4, 5, 7,)(8)

(1, 7,) (1, 8,)
(2, 3,) (2, 3,)
(4, 5, 6,)(8) (4, 5, 6, 7,)(9)

Our conclusion for $n=4$,

Consider $n=5$.

(1,)	(1,)	(1, 7,)
(2, 5,)	(2,)	(2, 3,)
(3, 4,)(6)	(3, 4, 5,)(6)	(4, 5, 6,)(8)

(1, 8,)
(2, 3,)
(4, 5, 6, 7,)(9)

Our conclusion for $n=5$.

Consider $n=6$.

(1,)	(1,)	(1, 8,)
(2, 5,)	(2,)	(2, 3,)
(3, 4,)(6)	(3, 4, 5,)(6)	(4, 5, 6, 7,)(9)

Our conclusion for $n=6$.

Now clearly for any $n\geq 7$ we shall continue to have only three possible combinations. Suppose we had $n\geq 7$, then if $x^1\in N_1$, $x^2\in N_2$, x^4 must be in N_3. Thus x^3 can be in N_2 or N_3. If $x^3\in N_2$, x^5 must be in N_3. Therefore x^8 cannot be in N_i, $(i=1, 2, 3)$. If $x^3\in N_3$ then x^6 cannot be in $N_i (i=1, 2, 3)$. This completes our proof.

4. Conjectures.

CONJECTURE 1. No semigroup S has a decomposition into the union of a finite number of disjoint n-th order niltants in S, for $n\geq 1$.

CONJECTURE 2. Let S be a topological semigroup. If $a\in S$ and $a^i\neq a$ for any integer $1<i\leq n+1$, n some positive integer, then there exists an *open* niltant of n-th order $N(a)$ in S containing a.

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REFERENCES

[1] A.H. Clifford and G.B. Preston: *The algebraic theory of semigroups*, Math. Surveys No. 7, Amer. Math. Soc., Providence, R. I., 1961.

[2] K. Iseki: *On mutants in semigroups*, Proc. Japan Acad. 38, pp. 478—479. 1962.

[3] J.B. Kim: *Mutants in semgiroups*, (Zech. Math. Journal, 19(94), Praha, pp.86—90, 1969.