ON DIMENSION OF HYPERSPACE OF A METRIC CONTINUUM

By Choon Jai Rhee

1. Introduction

The space $C(X)$ of all non-vacuous subcontinua of a metric continuum X with the Hausdorff metric has been investigated to a considerable extent. It is known that: X is Peanian if and only if $C(X)$ is Peanian [6] and [7]; $C(X)$ is always arcwise connected [1]: and if X is Peanian $C(X)$ is an absolute retract [9]: It is also known [3] that $C(X)$ is locally p-connected in the sense of Lefschetz for $p>0$, and the question of dimension is resolved there except for the case where X is non-Peanian. Recently it is shown [5] that if X is a pseudoarc in the plane E^2 which does not separate E^2, then $C(X)$ can be embedded in E^3. In this paper we will show that if X is a p-adic solenoid then the dimension of $C(X)$ is 2 and we will give some properties of $C(X)$ when X is a pseudoarc.

2. Dimension of $C(X)$.

Let S^1 be the unit circle in the complex plane. For each $n=1, 2, \cdots\cdots$ and a fixed integer $p>0$, let $X_n=S^1$ and $f_n(z)=z^p$ for $z \in S^1$. The p-adic solenoid is defined to be the inverse limit space of the inverse limit system $\{X_n, f_n\}$.

THEOREM. 2.1. *Let X be the p-adic solenoid. Then dim $C(X)=2$.*

PROOF. Let D be the set of all complex numbers w such that $|w|\leq 2\pi$. Since each subcontinuum of S^1 is a point, an arc, or S^1 itself, we define a function ϕ: $C(X)\to D$ by

$$\phi(A)=\begin{cases}(2\pi-r)z, & \text{if } A\neq S^1, \text{ where } z \text{ is the mid-point of } A \\ & \text{and } r \text{ is the length of } A. \\ \text{origin} & \text{if } A=S^1.\end{cases}$$

Then it is easy to see that ϕ is a homeomorphism of $C(X)$ onto the space D.

Let $f_n^*(A)=f_n(A)$ for $n=1, 2, \cdots\cdots$. Then each f_n^*: $C(X_{n+1})\to C(X_n)$ is continuous and the inverse limit space of the inverse limit system $\{C(X_n), f_n^*\}$ is homeomorphic to $C(X)$ [4], and hence dim $C(X)\leq 2$. On the other hand, since each f_n is a local homeomorphism, we can find arcs $A_n\subset X_n$ for which each restriction map $f_n|A_{n+1}$: $A_{n+1}\to A_n$ is a homeomorphism. The inverse limit space A of the inverse limit system $\{A_n, f_n|A_{n+1}\}$ is an arc in X. Since $C(A)$ is a 2-dimensional disk and $C(A)\subset C(X)$, we have dim $C(X)\geq 2$.

REMARK 2.2. Since each X_n is a topological group and each bonding map f_n is a homomorphism, it can be verified that each $C(X_n)$ is a topological semigroup whose product is defined by $AB=\{ab|a\in A,\ b\in B\}=BA$ and each induced map f_n^* is a homomorphism. Hence the inverse limit space of the system $\{C(X_n),\ f_n^*\}$ is a 2-dimensional abelian topological semigroup.

3. The Hyperspace of a Pseudoarc.

Let X be a compact metric space. It is possible to define [8] a real-valued continuous function μ on $C(X)$ with properties:

(i) If $A\subset B$ and $A\neq B$, then $\mu(A)<\mu(B)$

(ii) $\mu(X)=1$, and for each $x\in X$ $\mu(\{x\})=0$.

For convenience, we shall suppose throughout that μ is a certain fixed function with these properties.

The following four theorems can be found in [3].

3.1. If X is an indecomposable metric continuum and a_{AB} is an arc in $C(X)$ with $\bigcup\{D|D\in a_{AB}\}=X$, then $X\in a_{AB}$.

3.2. A metric continuum X is indecomposable if and only if $C(X)-X$ is not arcwise connected.

3.3. If X is a hereditarily indecomposable metric continuum, $A,\ B\in C(X)$, $A\cap B\neq\phi$, and $\mu(A)=\mu(B)$ then $A=B$.

3.4. A metric continuum X is hereditarily indecomposable if and only if $C(X)$ contains a unique arc between every pair of its elements.

Let X be a pseudoarc. Then X can be represented [2] as the inverse limit space of the inverse limit system $\{X_n,\ f_n\}$, where each X_n is the closed unit interval and $f_n=f_{n+1}$, $n=1,\ 2,\cdots\cdots$ is some suitable continuous map. Since each $C(X_n)$ is homeomorphic to the 2-simplex whose vertices are (0, 0), (1, 0), and (1, 1), we see that dim $C(X)\leq 2$.

THEOREM 3.5. *Let X be a pseudoarc. Then $C(X)$ is contractible.*

PROOF. It suffices to show [3] that the set $X_0^*=\{\{x\}|x\in X\}$ is contractible in

$C(X)$. Define $\Phi: X_0^* \times [0, 1] \to C(X)$ as follows: For each $(\{x\}, t) \in X_0^* \times [0, 1]$,

$$\Phi(\{x\}, t) = A_x, \text{ if } x \in A_x \in C(X) \text{ and } \mu(A) = t.$$

Then by 3.3 and 3.4, Φ is well defined. And $\Phi(\{x\}, 1) = X$, $\Phi(\{x\}, 0) = \{x\}$ for each $\{x\} \in X_0^*$.

Suppose that the sequence $\{(\{x_n\}, t_n)\}$ converges to $(\{x_0\}, t_0)$. Let $A_n = \Phi(\{x_n\}, t_n)$. We may assume without loss of generality that $\{x_n\} \to \{x_0\}$ and $t_n \to t_0$. If $\{A_{n_i}\}$ and $\{A_{n_j}\}$ are subsequences of $\{A_n\}$ which converges to A_0 and B_0 respectively, then it is easy to see that $x_0 \in A_0 \cap B_0$ and $t_0 = \mu(A_0) = \mu(B_0)$. Therefore, Φ is continuous.

THEOREM3. 6. *Let X be a pseudoarc. Then, for each neighborhood U of the element X in $C(X)$. There is a neighborhood V of X in $C(X)$ such that $V \subset U$ and the boundary of V is totally pathwise disconnected non-degenerated subcontinuum of $C(X)$.*

PROOF. Let $X_t^* = \Phi(X_0^*, t)$ $0 \leq t \leq 1$. Since X_0^* is homeomorphic to the continuum X, each X_t^* is a continuum. We will show that for a given U there is t_0 such that $V = \mu^{-1}(t_0, 1] \subset U$. We may note here that $X_t^* = \mu^{-1}(t)$.

First, assume that there is no t for which $X_t^* \subset U$. Then for each t, there is an element $A_t \in X_t^*$ such that $A_t \notin U$. We choose sequences $\{t_n\}$ and $\{A_{t_n}\}$ such that $\{t_n\}$ converges to 1 and $\{A_{t_n}\}$ converges to an element $A \in C(X)$. Then it is clear that $A = X$. Since $A \in U$, there is N such that $A_{t_n} \in U$ for all $n \geq N$. This is a contradiction.

Now let $t_0 < 1$ such that $\mu^{-1}(t_0) \subset U$. We may assume here that $U = \bigcap_{i=1}^{n} (O_i, W_i)$, where O_i and W_i are open sets in X. Let $B \in X_t^*$ for $t_0 < t \leq 1$, and $b \in B$. Then by 3.4, there is a unique are χ joining $\{b\}$ to B in $C(X)$ such that $\mu(\{b\}) = 0$ and $\mu(B) = t$. Then by the construction [3] of χ, the reis an element $A_0 \in X^*t_0$ such that $\mu(A) = t_0$, $b \in A_0$, and $A_0 \subset B$. Then by the definition of U and $A \in U$, we see that $B \in U$. Thus we have $\mu^{-1}(t_0, 1] \subset U$.

For each $0 \leq t < 1$, X_t^* is a totally pathwise disconnected non-degenerated continuum. Let $A \in X_t^*$, and $x \in X - \mathrm{A}$. Let χ be the unique arc in $C(X)$ joining $\{x\}$ to X. Then by 3.3 and 3.4, there is an element $B \in \chi$ such that $x \in B \in X_t^*$ and $A \cap B = \phi$. Hence X_t^* is a non-degenerated continuum. Suppose $\alpha: [0, 1] \to X_t^*$ is a path joining elements $A, B \in X_t^*$. Then there is an arc a_{AB} in $\alpha[0, 1] \subset X_t^*$ joining A to B. Assume that $A \neq B$. Let $C \in C(X)$ be the minimal element with respect to containing both A and B. Let a_{AC} and a_{BC} be arcs in $C(X)$ joining A to C and B to C respectively. Then if $D \in a_{AC} \cap a_{BC}$ then $D \supset A$ and $D \supset B$ by [3] so that $D = C$. Since a_{AB} is unique, $a_{AB} = a_{AC} \cup a_{BC} \subset X_t^*$. But $\mu(A) < \mu(C)$ so that $C \in X^*_t$. Therefore $A = B$. X_t^* is not pathwise connected.

Wayne State University

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