NOTES ON SUREACES OF CODIMENSION 2 IN A KAEHLERIAN MANIFOLD

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1. Introduction

For a Riemannian manifold admitting an infinitesimal special concircular transformation, we know the following Obata's theorem.

THEOREM A. [1] Let M be a complete connected Riemannian manifold of dimension $n(\geqq 2)$. In order that M admits a nontrivial solution of the system of differential equations

$$\nabla_\lambda \nabla_\kappa \phi + k\,\phi\, G_{\lambda\kappa} = 0, \quad k>0,$$

it is necessary and sufficient that M is isometric with a sphere S^n of radius $\frac{1}{\sqrt{k}}$ in the Euclidean $(n+1)$ space.

In this paper, We shall study several properties for surfaces in a Kaehlerian manifold by using of Theorem A.

2. Surfaces of codimension 2 in a Kaehlerian manifold

Let M be a surface of codimension 2 which is differentiably immersed in $\overline{M}$. We suppose that M is represented by equation

$$X^\lambda = X^\lambda(x^i)$$

in each coordinate neighborhood U of $\overline{M}$, $\{X^\lambda\}$ being coordinates defined in U and $\{x^i\}$ local coordinates defined in $M \cap U$.

On putting $g_{ji} = G_{\lambda\kappa} B_j{}^\lambda B_i{}^\kappa$ we see that g_{ji} define in M a Riemannian metric which is called the induced metric, where $B_i{}^\lambda = \partial X^\lambda / \partial x^i$.

The Kaehlerian manifold $\overline{M}$ being orientable, we assume that the surface M is also orientable and that $B_1{}^\lambda, \cdots, B_{2n-2}{}^\lambda$ are chosen in such a way that they form a frame of positive orientation. We then choose two loca fields of mutually orthogonal unit vectors C^λ and D^λ in such a way that $C^\lambda, D^\lambda, B_1{}^\lambda, \cdots, B_{2n-2}{}^\lambda$ form a frame of positive orientation in $\overline{M}$. If $'C^\lambda$ and $'D^\lambda$ are another set of normals satisfying the same condition, then we know

$$(2.1) \qquad 'C^\lambda = \cos\theta\, C^\lambda - \sin\theta\, D^\lambda, \quad 'D^\lambda = \sin\theta\, C^\lambda + \cos\theta\, D^\lambda.$$

And then we find

$$(2.2)\qquad \begin{aligned} &G_{\lambda\kappa}B_i^\lambda C^\kappa=G_{\lambda\kappa}B_i^\lambda D^\kappa=G_{\lambda\kappa}C^\lambda D^\kappa=0,\\ &G_{\lambda\kappa}C^\lambda C^\kappa=G_{\lambda\kappa}D^\lambda D^\kappa=1,\\ &B_j^\lambda B^i{}_\lambda=\delta_j^i,\ \ B_j^\lambda B^j{}_\mu=\delta_\mu^\lambda-C_\mu C^\lambda-D_\mu D^\lambda,\end{aligned}$$

where we have put $B^j{}_\kappa=G_{\lambda\kappa}B_i^\lambda g^{ji}$, $(g^{ji})=(g_{ji})^{-1}$, $C_\kappa=G_{\lambda\kappa}C^\lambda$, $D_\kappa=G_{\lambda\kappa}D^\lambda$. Therefore, we can put

$$(2.3)\qquad \begin{aligned} &F_\lambda{}^\kappa B_i^\lambda=f_i^jB_j^\kappa+f_iC^\kappa+g_iD^\kappa,\\ &F_\lambda{}^\kappa C^\lambda=-f^iB_i^\kappa+fD^\kappa,\ \ F_\lambda{}^\kappa D^\lambda=-g^iB_i^\kappa-fC^\kappa,\end{aligned}$$

f^i and g^i being defined by $f^i=g^{ij}f_j$ and $g^i=g^{ij}g_j$ respectively. From (2.2) and (2.3) we get

$$(2.4)\qquad \begin{aligned} &f_i^j=B_i^\lambda F_\lambda{}^\kappa B^j{}_\kappa.\\ &f_i=B_i^\lambda F_\lambda{}^\kappa C_\kappa,\ \ g_i=B_i^\lambda F_\lambda{}^\kappa D_\kappa,\\ &f=C^\lambda F_\lambda{}^\kappa D_\kappa.\end{aligned}$$

Denoting by H_{ji} and K_{ji} the second fundamental tensor of the surface M with respect to the normals C^λ and D^λ and putting

$$H^j{}_i=g^{jh}H_{hi},\ \ K^j{}_i=g^{jh}K_{hi}$$

then the Gauss and the Weingarten equations for M are given respectively by

$$(2.5)\qquad \begin{aligned} &\nabla_jB_i^\lambda=H_{ji}C^\lambda+K_{ji}D^\lambda,\\ &\nabla_jC^\lambda=-H_j{}^iB_i^\lambda+L_jD^\lambda,\ \ \nabla_jD^\lambda=-K_j{}^iB_i^\lambda-L_jC^\lambda.\end{aligned}$$

Differentiating covariantly the both sides of (2.4) and taking account of (2.5), we find

$$(2.6)\qquad \begin{aligned} &\nabla_jf_i^h=f_iH_j^h+g_iK_j^h,\\ &\nabla_jf_i=-fK_{ji}+g_iL_j-f_i^hH_{jh},\\ &\nabla_jg_i=H_{ji}-f_i^jK_{jh}-f_iL_j,\\ &\nabla_jf=K_{ji}f^i-H_{ji}g^i.\end{aligned}$$

Transvecting again the both sides of (2.3) with $F_\lambda{}^K$ and making use of (2.3), we obtain

$$(2.7)\qquad \begin{aligned} &f_i^hf_h^j=-\delta_i^j+f_if^j+g_ig^j,\\ &f_i^hf_h=fg_i,\ \ f_i^hg_h=-ff_i,\\ &f^if_i=g^ig_i=1-f^2,\ \ f^ig_i=0.\end{aligned}$$

Last, we denote by $\bar R_{\nu\mu\lambda\kappa}$ and R_{kjih} the components of the curvature tensors of $\bar M$ and M respectively, then we find

(2.8) $R_{kji}{}^h = H_k{}^h H_{ji} - H_{ki} H_j{}^h + K_k{}^h K_{ji} - K_{ki} K_j{}^h + \bar{R}_{\nu\mu\lambda}{}^\kappa B_k{}^\nu B_j{}^\mu B_i{}^\lambda B^h{}_\kappa$,

$\nabla_k H_{ji} - \nabla_j H_{ki} + K_{ki} L_j - K_{ji} L_k = \bar{R}_{\nu\mu\lambda}{}^\kappa B_k{}^\nu B_j{}^\mu B_i{}^\lambda C_\kappa$,

$\nabla_k K_{ji} - \nabla_j K_{ki} - H_{ki} L_j + H_{ji} L_k = \bar{R}_{\nu\mu\lambda}{}^\kappa B_k{}^\nu B_j{}^\mu B_i{}^\lambda D_\kappa$,

Which are the so-called Gauss and Codazzi equations.

First, We shall prove the following

LEMMA 2.1 *The scalar function f defined by* (2.1) *is determined independently of the choice of mutually orthogonal unit normal vectors C^λ and D^λ to the surface M, and consequently f is globally defined in M.*

PROOF. Let $'C^\lambda$ and $'D^\lambda$ be mutually orthogonal unit normal vectors to the manifold M at a point P, then we find that, between a pair of unit normal vectors (C^λ, D^λ) and $('C^\lambda, 'D^\lambda)$ chosen as above at each point of M the relatons (2.1) hold. So we find, $'f = 'C^\lambda F_\lambda{}^\kappa {}'D_\kappa$, which shows that f is independent of the choice of unit normal vectors C^λ and D^λ and that f is a globally defined.

3. Totally umbilical surfaces of codimension 2 in a Kaehlerian manifold.

When, at each point of the surface M of codimension 2, the relations $H_{ji} = Hg_{ji}$, $K_{ji} = Kg_{ji}$ are always valid, the surface is called a totally umbilical surface, H and K being given by $\frac{1}{2n-2} g^{ji} H_{ji}$, $\frac{1}{2n-2} g^{ji} K_{ji}$ respectively.

The mean curvature vector field H^λ of M in $\bar{M}$ is given by

$$H^\lambda = HC^\lambda + KD^\lambda$$

Then the following theorem is well known [4]

THEOREM B. *Let M be a $(2n-1)$-dimensional totally umbilical surface in a $(2n+1)$-dimensional Riemannian manifold $\bar{M}$. If the covariant derivative $\nabla_j H^\lambda$ of the mean curvature vector field H^λ of M is tangent to M, then M is of constant mean curvature.*

Next, We shall prove

LEMMA 3.1. *Let M be a $(2n-2)$-dimensional totally umbilical surface with non-zero mean curvature in a Kaehlerian manifold. Suppose that $\nabla_j H^\lambda$ is tangent to M, then the function f defined by* (2.4) *is non-constant.*

PROOF. Suppose that the function f is constant in M. Differentiating covariantly the last equation of (2.6), we get

(3.1) $$\nabla_j\nabla_i f=-f(H^2+K^2)g_{ji},$$

from which we have $f=0$. Again, transvecting $\nabla_j f$ of (2.6) with f^j and g^j respectively and taking account of (2.7) and $f=0$, we find $K=0$, $H=0$ respectively. These results contradict to our assumption. Thus, the function f is non-constant,

As a consequence of this lemma and theorem B, we have

LEMMA 3.2 *Let M be a $(2n-2)$-dimensional umbilical surface with non-zero mean curvature in a Kaehlerian manifold. Suppose that $\nabla_j H^\lambda$ is tangent to M, then the gradient of the scalar function f is an infinitesimal special concircular transformation.*

Combining lemma 3.2 and theorem A, we have

THEOREM 3.3. *Let M be a $(2n-2)$-dimensional complete connected totally umbilical surface with non-zero mean curvature in Kaehlerian manifold $(n\geq 2)$. Suppose that $\nabla_j H^\lambda$ is tangent to M, then M is isometric with a sphere of radius $\frac{1}{\sqrt{H^2+K^2}}$ in the Euclidean space, where H^2+K^2 is the mean curvature of M.*

It has been proved that the Kaehlerian manifold of constant holomorphic sectional curvature $\overline{K}$ has the curvature tensor of the form [2]

(3.2) $$R_{\nu\mu\lambda\kappa}=k(G_{\nu\kappa}G_{\mu\lambda}-G_{\nu\lambda}G_{\mu\kappa}+F_{\nu\kappa}F_{\mu\lambda}-F_{\nu\lambda}F_{\mu\kappa}-2F_{\nu\mu}F_{\lambda\kappa}),$$ where

$k=\overline{K}/4$ is constant. Substituting (3.2) into (2.8),

(3.3) $$R_{kjih}=H_{kh}H_{ji}-H_{ki}H_{jh}+K_{kh}K_{ji}-K_{ki}K_{jh}+k(g_{kh}g_{ji}-g_{ki}g_{jh}+f_{kh}f_{ji}-f_{ki}f_{jh}-2f_{kj}f_{ih})$$

and

(3.4) $$\nabla_k H_{ji}-\nabla_j H_{ki}-K_{ji}L_k+K_{ki}L_j=k(f_k f_{ji}-f_j f_{ki}-2f_i f_{kj}),$$

f_{ji} being defined by $f_{ji}=g_{ih}f_j{}^h$. Suppose that M is a totally umbilical suaface of codimension 2 with non-zero mean curvature in $\overline{M}$ and that the covariant derivative of the mean curvature vector of M is tangent to M. Transvecting (3.4) with g^{ji}, we get $0=-3kf(1-f^2)$ by virtue of the skew symmetry of f_{ji}. Taking account of lemma 3.1 and transvecting (3.3) with g^{kh}, we obtain $R_{ji}=(2n-3)(H^2+K^2)g_{ji}$. Thus we have the following

THEOREM 3.4 *Let M be a totally umbilical surface of codimension 2 with non-zero mean curvature in a Kaehlerian manifold of constant holomorphic sectional cucvature. If the covariant derivative of the mean curvature vector of M is tangent* *to M, then M is Einstein space.*

From this theorem, we have

COROLLARY 3.5. *If the covariant derivative of the mean curvature vector of M is tangent to M, then there is no totally umbilical surface of codimension 2 with non-zero mean curvature other than Einstein in a Kaehlerian manifold of constant holomorphic sectional curvature.*

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