EXPANSION OF THE H-FUNCTION INVOLVING GENERALIZED LEGENDRE ASSOCIATED FUNCTIONS AND H-FUNCTIONS

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1. Introduction.

In this paper, an integral involving H-function and a generalized Legendre associated function has been evaluated and it has been employed to establish an expansion of the H-function, in a series of the products of H-functions and generalized Legendre associated functions. The generalized Legendre associated functions reduce to associated Legendre functions on setting $m=n$ and to Legendre functions on taking $m=n=0$. Also, a large number of special functions are particular cases of the H-function. So, on specializing the parameters of these functions in the expansion, we may get many new as well as known expansions.

In [7], Kuipers and Meulenbeld have defined generalized Legendre associated functions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ as two linearly independent solutions of the differential equation

$$(1.1)\qquad (1-z^2)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+\left\{k(k+1)-\frac{m^2}{2(1-z)}-\frac{n^2}{2(1+z)}\right\}w=0,$$

at all points of the z-plane in which a cross-cut exists along the real axis from 1 to $-\infty$ and in [8], these functions have been defined for the real values of z on the cross-cut for $-1<z<1$.

The H-function has been introduced by Fox [5, p. 408] and its conditions of validity, asymptotic expansions and analytic continuations have been discussed by Braaksma [1]. Following the definition given by Braaksma [1, p. 239−241], it will be represented as follows:

$$(1.2)\qquad H_{p,q}^{m,n}\left[z\left|\begin{matrix}\{(a_p,\ \alpha_p)\}\\ \{(b_q,\ \beta_q)\}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_T\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\xi)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi)}z^{\xi}d\xi,$$

where $\{(f_r,\ \gamma_r)\}$, stand for the set of the parameters $(f_1,\ \gamma_1),\ \cdots,\ (f_r,\ \gamma_r)$.

2. The integration.

The integration to be established is

$$(2.1)\qquad \int_{-1}^{1}(1-x)^{\rho}(1-x^{2})^{\frac{n}{2}}P_{k}^{m,n}(x)\ H_{r,s}^{l,u}\left[z(1-x)^{\delta}\,\middle|\,\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right]dx$$

$$=\frac{2^{\rho+\frac{3n-m}{2}+1}\,\Gamma(k+\frac{m+n}{2}+1)}{\Gamma\left(k-\frac{m+n}{2}+1\right)}$$

$$\times H_{r+2,s+2}^{l+1,u+1}\left[2^{\delta}z\,\middle|\,\begin{matrix}(-\rho-\frac{m+n}{2},\delta),\{(a_r,\alpha_r)\},(-\rho+\frac{m-n}{2},\delta)\\ (k-\rho-n,\delta),\{(b_s,\beta_s)\},(-1-\rho-k-n,\delta)\end{matrix}\right],$$

where k, m *and* $k-\frac{m+n}{2}$ *are non-negative integers,* $\delta>0$, $\mathrm{Re}\left(1+\rho+\frac{n}{2}+\delta b_h/\beta_h\right)>0$, $(h=1,2,\cdots,l)$, $\sum_{1}^{s}(\beta_j)-\sum_{1}^{r}(\alpha_j)\geq 0$, $\sum_{1}^{u}(\alpha_j)-\sum_{u+1}^{r}(\alpha_j)+\sum_{1}^{l}(\beta_j)-\sum_{l+1}^{s}(\beta_j)\equiv\phi>0$ *and* $|\arg z|<\frac{1}{2}\phi\pi$.

PROOF. Expressing the H-function in the integrand as Mellin-Barnes type integral (1.2), interchanging the order of integration, which is justifiable due to the absolute convergence of the integrals involved in the process, we have

$$(2.2)\qquad \frac{1}{2\pi i}\int_{T}\frac{\prod_{j=1}^{l}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{u}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=l+1}^{s}\Gamma(1-b_j+\beta_j\xi)\prod_{j=u+1}^{r}\Gamma(a_j-\alpha_j\xi)}z^{\xi}$$

$$\times\int_{-1}^{1}(1-x)^{\rho+\frac{n}{2}+\delta\xi}(1+x)^{\frac{n}{2}}P_{k}^{m,n}(x)\,dx\,d\xi.$$

Evaluating the inner integral with the help of the result [6, p. 149(6)], i.e.,

$$(2.3)\qquad \int_{-1}^{1}(1-x)^{\rho}(1+x)^{\frac{n}{2}}P_{k}^{m,n}(x)\,dx$$

$$=\frac{2^{\rho-\frac{m}{2}+n+1}\Gamma\left(\rho+\frac{m}{2}+1\right)\Gamma\left(k-\rho-\frac{n}{2}\right)\Gamma\left(k+\frac{m+n}{2}+1\right)}{\Gamma\left(k-\frac{m+n}{2}+1\right)\Gamma\left(-\rho+\frac{m}{2}\right)\Gamma\left(k+\rho+\frac{n}{2}+2\right)},$$

where $\mathrm{Re}(\rho)>-1$; and applying (1.2), the definition of the H-function, we get the result (2.1).

3. Expansion Formula.

Assumptions:

(i) δ is a positive number and l,u,r,s are integers such that $1\leq l\leq s$, $0\leq u\leq r$.

(ii) $\sum_1^s(\beta_j)-\sum_1^r(\alpha_j)>0$ when $z\neq 0$ and if $\sum_1^s(\beta_j)-\sum_1^r(\alpha_j)=0$,

then $0<|z|<\prod_{j=1}^r(\alpha_j)^{-\alpha_j}\prod_{j=1}^s(\beta_j)^{\beta_j}$.

(iii) $|\arg z|<\frac{1}{2}\phi\pi$ where $\phi=\left[\sum_1^u(\alpha_j)-\sum_{u+1}^r(\alpha_j)+\sum_1^l(\beta_j)-\sum_{l+1}^s(\beta_j)\right]>0$.

(iv) m is a non-negative integer and $\mathrm{Re}\left(1+\rho+\frac{n}{2}+\delta b_h/\beta_h\right)>0$, $(h=1,2,\cdots,l)$.

and

(v) $\alpha_i(b_h+\nu)\neq\beta_h(a_i-\eta-1)$, $\delta(b_h+\nu)\neq-\beta_h(\rho+\frac{m+n}{2}+\eta+1)$,

$(\nu,\ \eta=0,1,\cdots;\ h=1,2,\cdots,l;\ i=1,2,\cdots,u)$.

Then

$$(3.1)\qquad (1-x)^\rho(1-x^2)^{\frac{n}{2}}H_{r,s}^{l,u}\left[z(1-x)^\delta\left|\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right.\right]$$

$$=2^{\rho+\frac{m+n}{2}}\sum_{N=0}^\infty\frac{(1+2N)\Gamma\left(1+N-\frac{m-n}{2}\right)}{\Gamma\left(1+N+\frac{m-n}{2}\right)}P_N^{m,n}(x)$$

$$\times H_{r+2,s+2}^{l+1,u+1}\left[2^\delta z\left|\begin{matrix}(-\rho-\frac{m+n}{2},\delta),\ \{(a_r,\alpha_r)\},\ (-\rho+\frac{m-n}{2},\delta)\\ (N-\rho-n,\delta),\ \{(b_s,\beta_s)\},\ (-1-\rho-N-n,\delta)\end{matrix}\right.\right].$$

PROOF. Let

$$(3.2)\qquad (1-x)^\rho(1-x^2)^{\frac{n}{2}}H_{r,s}^{l,u}\left[z(1-x)^\delta\left|\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right.\right]=\sum_{N=0}^\infty C_N P_N^{m,n}(x).$$

Equation (3.2) is valid, since the expression on the left hand side is continuous and of bounded variation in the interval $(-1,1)$.

Now, multiplying both the sides of (3.2) by $P_k^{m,n}(x)$ and integrating with respect to x from -1 to 1: on the right hand side changing the order of summation and integration in view of [2, p.176(75)], we get

$$(3.3)\qquad \int_{-1}^1(1-x)^\rho(1-x^2)^{\frac{n}{2}}P_k^{m,n}(x)\ H_{r,s}^{l,u}\left[z(1-x)^\delta\left|\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right.\right]dx$$

$$=\sum_{N=0}^\infty C_N\int_{-1}^1 P_k^{m,n}(x)\ P_N^{m,n}(x)\ dx.$$

On the left hand side using (2.1) and on the right hand side applying orthogonality property of the generalized Legendre associated functions, i.e.

$$(3.4)\qquad \int_{-1}^{1} P_k^{m,n}(x)\, P_N^{m,n}(x)\, dx=0,\ k\neq N,$$

and

$$(3.5)\qquad \int_{-1}^{1}\left[P_k^{m,n}(x)\right]^2 dx=\frac{2^{1-m+n}\Gamma\left(1+k+\frac{m-n}{2}\right)\Gamma\left(1+k+\frac{m+n}{2}\right)}{(1+2k)\Gamma\left(1+k-\frac{m+n}{2}\right)\Gamma\left(1+k-\frac{m-n}{2}\right)},$$

which can be easily obtained on using [6, p. 149(4)] in [3, p. 285(9 & 5)]; we get

$$(3.6)\qquad C_k=2^{\rho+\frac{m+n}{2}}\frac{(1+2k)\Gamma\left(1+k-\frac{m-n}{2}\right)}{\Gamma\left(1+k+\frac{m-n}{2}\right)}$$

$$\times H_{r+2,\ s+2}^{l+1,\ u+1}\left[2^{\delta}z\left|\begin{matrix}(-\rho-\frac{m+n}{2},\ \delta),\ \{(a_r,\ \alpha_r)\},\ (-\rho+\frac{m-n}{2},\ \delta)\\ (k-\rho-n,\ \delta),\quad \{(b_s,\ \beta_s)\},\quad (-1-\rho-k-n,\ \delta)\end{matrix}\right.\right].$$

Using (3.6) in (3.2), we get the result (3.1). We have so far shown that (3.1) is a formal identity. We now prove that (3.1) is defined and converges under the stated hypothesis.

The necessary conditions to ensure the convergence and meaning of the H-functions are covered in (i) to (iii) and (v). The remaining assumptions arise from the consideration of the convergence of the infinite series in (3.1).

On using [1, p. 279(6.4)], i.e.

$$(3.7)\qquad H_{p,q}^{m,n}\left[z\left|\begin{matrix}\{(a_p,\ \alpha_p)\}\\ \{(b_q,\ \beta_q)\}\end{matrix}\right.\right]=\sum_{h=1}^{\infty}\sum_{\nu=0}^{\infty}\frac{\prod_{1}^{m}{}' \Gamma(b_j-\beta_j(b_h+\nu)/\beta_h)}{\prod_{m+1}^{q}\Gamma(1-b_j+\beta_j(b_h+\nu)/\beta_h)}$$

$$\times\frac{\prod_{1}^{n}\Gamma(1-a_j+\alpha_j(b_h+\nu)/\beta_h)(-1)^{\nu}z^{(b_h+\nu)/\beta_h}}{\prod_{n+1}^{p}\Gamma(a_j-\alpha_j(b_h+\nu)/\beta_h)\ \nu!\ \beta_h},$$

where $\prod_{1}^{m}{}'$ means the product of the factors with $j=1,\ \cdots\cdots,\ j=m$ save $j=h$; and

$$(3.8)\qquad \frac{\Gamma(a+n)}{\Gamma(b+n)}=n^{a-b}[1+0(n^{-1})],$$

we have

$$(3.9)\qquad H_{r+2,\ s+2}^{l+1,\ u+1}\left[2^{\delta}z\left|\begin{matrix}(-\rho-\frac{m+n}{2},\ \delta),\ \{(a_r,\ \alpha_r)\},\ (-\rho+\frac{m-n}{2},\ \delta)\\ (N-\rho-n,\delta),\ \{(b_s,\ \beta_s)\},\ (-1-\rho-N-n,\ \delta)\end{matrix}\right.\right]$$

$$\times 0(N^{-2-2\rho-2n-2\delta b_h/\beta_h}),$$

where $h=1,\ 2,\ \cdots,\ l$.

Now, applying [9, p. 32(9)], [4, p. 399(2.5)] and (3.8) to [7, p. 441(12)], we get

(3.10) $$P_N^{m,n}(x)\sim 0\ (N^{-1+m}).$$

In view of (3.8), (3.9) and (3.10), it is easy to see that the series on the right hand side of (3.1) converges when m is a positive integer and

$\mathrm{Re}\left(1+\rho+\frac{n}{2}+\delta b_h/\beta_h\right)>0,\ (h=1,2,\cdots,l).$

This completes the proof of the expansion formula.

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